Prove that the following sets are commutative rings with unity.

1. $R = \{0, 2, 4, 6, 8\}$,under addition and multiplication mod 10.

2. $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$, under ordinary addition and multiplication of real numbers.

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Remember:

1. An *integral domain* is a commutative ring (with 1), with no zero-divisors (i.e. such that if $ab = 0$, then $a = 0$ or $b = 0$.

2. A *field* is a commutative ring (with 1), in which every non-zero element is a unit (i.e. in which every non-zero element has a multiplicative inverse).

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1. We know that if a commutative ring (with 1) has no zero-divisors, then the cancellation property holds. That is, if a commutative ring (with 1) has no zero-divisors, then *if* $ab = ac$, *then* $b = c$.

   Show that the converse is also true; that is, show that a commutative ring with the cancellation property has no zero-divisors.

2. Show that every nonzero element of $\mathbb{Z}_n$ is either a unit or a zero-divisor.

3. Find a non-zero element in a ring that is neither a zero-divisor nor a unit.

   Hint: Consider the ring $\mathbb{Z}[x]$, the set of all polynomials with integers as coefficients.