Definition: A subgroup H of a group G is a **normal subgroup** iff $aH = Ha$ for all $a \in G$. We denote a normal subgroup by $H \lhd G$.

Theorem 9.1: A subgroup $H \ \leq \ G$ is normal $\iff$ $(x)^{-1}Hx \subseteq H$ for all $x \in G$.

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Let $H = \{\epsilon, \begin{bmatrix}1 & 2 & 3\end{bmatrix}, \begin{bmatrix}1 & 3 & 2\end{bmatrix}\}$. We've seen that $H \lhd S_3$.
We let

$$\begin{aligned} S_H &= \text{ the set of all left cosets of } H \text{ in } S_3 \\ &= \{\alpha H | \alpha \in S_3\} \\ &= \{H, \begin{bmatrix}1 & 2\end{bmatrix} H\}, \end{aligned}$$

where

$$H = \epsilon H = \begin{bmatrix}1 & 2 & 3\end{bmatrix} H = \begin{bmatrix}1 & 3 & 2\end{bmatrix} H$$
$$\begin{bmatrix}1 & 2\end{bmatrix} H = \begin{bmatrix}1 & 3\end{bmatrix} H = \begin{bmatrix}2 & 3\end{bmatrix} H$$

We defined an operation $*$ on the elements of the set S_H as follows: For any $\alpha H, \beta H \in S_H$,

$$\alpha H * \beta H \stackrel{def}{=} (\alpha \circ \beta)H.$$

Question: Is this operation **well-defined**? That is, if $\alpha_1 H = \alpha_2 H$ and $\beta_1 H = \beta_2 H$, then must $\alpha_1 H * \beta_1 H = \alpha_2 H * \beta_2 H$?

Let $K = \{\epsilon, \begin{bmatrix} 1 & 2 \end{bmatrix}\}$. We've seen that $K \ntriangleleft S_3$.

We let

$$\begin{aligned} S_K &= \text{ the set of all left cosets of } K \text{ in } S_3 \\ &= \{\alpha K | \alpha \in S_3\} \\ &= \{K, \begin{bmatrix} 1 & 3 \end{bmatrix} K, \begin{bmatrix} 2 & 3 \end{bmatrix} K\}. \end{aligned}$$

Question: If we use the same idea as before to define an operation $*$ on the elements of the set S_K: For any $\alpha K, \beta K \in S_K$, define

$$\alpha K * \beta K \stackrel{def}{=} (\alpha \circ \beta) K,$$

is $*$ well-defined on S_K?