Recall: The Fundamental Theorem of Finite Abelian Groups:

Any finite Abelian group can be written as

$$\mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \ldots \oplus \mathbb{Z}_{p_k^{n_k}},$$

where the p_i are not necessarily distinct, and where $|G| = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$.

In Class Work

1. Find all possible groups of order 270, up to isomorphism class. Which of your isomorphism classes is $\mathbb{Z}_{54} \oplus \mathbb{Z}_5$ in?
2. What is the isomorphism class of $U(16)$? What is the isomorphism class of $U(20)$? What might you conjecture from these results?

Solutions:

1. Find all possible groups of order 270, up to isomorphism class. Which of your isomorphism classes is $\mathbb{Z}_{54} \oplus \mathbb{Z}_5$ in?
   - $270 = 2 \times 3^3 \times 5 \quad = 2 \times 3^2 \times 3 \times 5 \quad = 2 \times 3 \times 3 \times 3 \times 5$ and so the isomorphism classes are represented by:

     $$\begin{aligned} G &\approx \mathbb{Z}_2 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5 \\ \textbf{or } G &\approx \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \\ \textbf{or } G &\approx \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5. \end{aligned}$$

     Very cool! There are only 3 possible groups (up to isomorphism) of Abelian groups of order 270!
   - Since $\mathbb{Z}_{54} \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{27}$ (because we know that if $gcd(m, n) = 1$, then $\mathbb{Z}_m \oplus \mathbb{Z}_n \approx \mathbb{Z}_{mn}$), we know

     $$\mathbb{Z}_{54} \oplus \mathbb{Z}_5 \approx \mathbb{Z}_{27} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5.$$

2. What is the isomorphism class of $U(16)$? What is the isomorphism class of $U(20)$? What might you conjecture from these results?
   - What is the isomorphism class of $U(16)$?

$$U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\} \qquad \text{so} \qquad |\,U(16)\,| = 8 = 2^3.$$

Possible isomorphism classes for $U(16)$: $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

To be isomorphic to $\mathbb{Z}_8$, $U(16)$ must have an element of order 8.
To be isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, it must have only elts of order 1,2, or 4.
To be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, it must have elts only of order 1 or 2.

$\lvert 1 \rvert = 1$	$\lvert 9 \rvert = 2$
$\lvert 3 \rvert = 4$	$\lvert 11 \rvert = 4$
$\lvert 5 \rvert = 4$	$\lvert 13 \rvert = 4$
$\lvert 7 \rvert = 2$	$\lvert 15 \rvert = 2$

Thus $U(16) \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

2.

- What is the isomorphism class of $U(20)$?
  This group has the same order, and so has the same possibilities.

  $$\begin{aligned} |1| &= 1 & \qquad |11| &= 2 \\ |3| &= 4 & \qquad |13| &= 4 \\ |7| &= 4 & \qquad |17| &= 4 \\ |9| &= 2 & \qquad |19| &= 2 \end{aligned}$$

  Since again every element has order 2 or 4, $U(20)$ must be in the same isomorphism class as $U(16)$ is.
- What might you conjecture from these results?
  It seems as if, when given a specific finite Abelian group G, we find which of the possible isomorphism classes it belongs to by comparing the orders of the elements. This would lead me to conjecture that if two finite Abelian groups have the same number of elements of each order, they are isomorphic.

In Class Work

1. Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10. Is the same thing true if the order of a finite Abelian group is divisible by 4?

Solution:

1. Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10. Is the same thing true if the order of a finite Abelian group is divisible by 4?

   Because 10 divides the order of the group, we know from the corollary that there is a subgroup of order 10. Since all groups of order 10 are cyclic (we know this from the FToAG), there must be a cyclic subgroup of order 10.

   If the order is divisible by 4, however, everything changes. We know that there must be a subgroup of order 4, but we don't know whether that subgroup must be cyclic or not.