Definition:

Let $G_1, G_2, \ldots, G_n$ be groups. The **external direct product** $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ is the set of n tuples with component-wise operation. That is,

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n = \{(g_1, \ldots, g_n) | g_i \in G_i\}$$

and

$$(g_1, g_2, \ldots, g_n) \cdot (g_1', g_2', \ldots, g_n') = (g_1 g_1', g_2 g_2', \ldots, g_n g_n'),$$

where $g_i g_i'$ represents the operation of G_i.

Example:

List the elements of $U(4) \oplus \mathbb{Z}_2 \oplus D_1$.

$$\begin{aligned} U(4) &= (\{1,3\}, \cdot \mod 4) \\ \mathbb{Z}_2 &= (\{0,1\}, + \mod 2) \\ D_1 &= (\{R_0, h\}, \circ) \end{aligned}$$

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$$\begin{aligned} U(4) \oplus \mathbb{Z}_2 \oplus D_1 &\stackrel{\text{def}}{=} \{(g_1, g_2, g_3) | g_1 \in U(4), g_2 \in \mathbb{Z}_2, g_3 \in D_1\} \\ &= \{(1,0,R_0), (1,1,R_0), (3,0,R_0), (3,1,R_0), \\ &\quad (1,0,h), (1,1,h), (3,0,h), (3,1,h)\} \end{aligned}$$

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Example:

List the elements of $U(4) \oplus \mathbb{Z}_2 \oplus D_1$.

$$\begin{aligned} U(4) \oplus \mathbb{Z}_2 \oplus D_1 &\stackrel{\text{def}}{=} \{(g_1, g_2, g_3) | g_1 \in U(4), g_2 \in \mathbb{Z}_2, g_3 \in D_1\} \\ &= \{(1,0,R_0), (1,1,R_0), (3,0,R_0), (3,1,R_0), \\ &\quad (1,0,h), (1,1,h), (3,0,h), (3,1,h)\} \end{aligned}$$

The operation is component-wise. For example,

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Inverses also turn out to be component-wise. For example:

$$(1,1,h)^{-1} = (1^{-1}, 1^{-1}, h^{-1}) = (1,1,h).$$

Recall:

- $G_1 \oplus \cdots \oplus G_n$ is a group.

- **Theorem 8.1:** Let $G_1, \ldots, G_n$ be groups, and let $g_i \in G_i$. Then

$$|(g_1, \ldots, g_n)| = \text{lcm}\{|g_1|, \ldots, |g_n|\}.$$

In Class Work

1. Decide whether the following pairs of groups are isomorphic or not.
   (a) $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_{27}$
   (b) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_{30}$
2. Prove that $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
3. Prove or disprove that $\mathbb{Z} \oplus \mathbb{Z}$ is a cyclic group.

Solutions:

1. Decide whether the following pairs of groups are isomorphic or not.

   (a) $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_{27}$

   While it is true that $27 = 3 \cdot 9$, 3 and 9 are not relatively prime. Thus these two groups are not isomorphic.

   (b) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_{30}$

   Since $30 = 2 \cdot 3 \cdot 5$ and since any pair of 2, 3, and 5 are relatively prime, these two groups *are* isomorphic.

Solutions:

2. Prove that $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

   Both groups have order 8.

   Every non-identity element of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has order 2.

   However, in $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, the element $(1, 0)$ has order 4. Since isomorphisms preserve order, there can not be an isomorphism between these two groups.

3. Prove or disprove that $\mathbb{Z} \oplus \mathbb{Z}$ is a cyclic group.

Do there exist m and n so that

$$< (m, n) >= \mathbb{Z} \oplus \mathbb{Z}?$$

If $m = 0$, then $(1, 0) \notin < (0, n) >$, so $< 0, n > \neq \mathbb{Z} \oplus \mathbb{Z}$.

Similarly, $< (m, 0) > \neq \mathbb{Z} \oplus \mathbb{Z}$, and it's quite obvious that $< (0, 0) > \neq \mathbb{Z} \oplus \mathbb{Z}$. Thus it's only possible if $m \neq 0$, $n \neq 0$.

$$\begin{aligned} < (m, n) > &= \{\ldots, (-3m, -3n), (-2m, -2n), (-m, -n), \\ &\quad (0, 0), (m, n), (2m, 2n), (3m, 3n), \ldots\}. \end{aligned}$$

Since $m \neq 0, n \neq 0$, then the elements $(m, 0)$ and $(0, n)$ aren't in $< (m, n) >$, so $< (m, n) > \neq \mathbb{Z} \oplus \mathbb{Z}$.

Thus $\mathbb{Z} \oplus \mathbb{Z}$ is *not* cyclic.