Recall:

Definition: A subgroup H of a group G is a **normal subgroup** iff $aH = Ha$ for all $a \in G$. We denote a normal subgroup by $H \triangleleft G$.

Theorem 9.1: A subgroup $H \leq G$ is normal $\iff x^{-1}Hx \subseteq H$ for all $x \in G$.

Recall:

We saw quickly last Wednesday that sometimes the set of all left (or right) cosets form a group and sometimes they don't:

Examples:

- The set of all left cosets of $n\mathbb{Z}$ in $\mathbb{Z}$ forms a group. Specifically, the set

  $$\{0 + n\mathbb{Z}, 1 + n\mathbb{Z}, 2 + n\mathbb{Z}, \dots, n - 1 + n\mathbb{Z}\}$$

  is a group under the operation $(a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a + b) + n\mathbb{Z}$.

- The set of all left cosets of $< \begin{pmatrix} 1 & 2 \end{pmatrix} >$ in S_3 do *not* form a group under the analogous operation $\alpha H \circ \beta H = (\alpha\beta)H$, as this operation isn't even well-defined.

Question:

When does the set of all left (or right) cosets of a group G form a group under the operation

$$aG * bG \stackrel{def}{=} (a * b)G?$$

Solutions to Problem 3 from Monday:

3. (a) **Let H** $=< \begin{pmatrix}1 & 2 & 3\end{pmatrix} >$ **in** $\mathbf{S_3}$**, and consider the left cosets of** H**,**

$$\mathbf{S_H} = \{\varepsilon \mathbf{H}, \begin{pmatrix}1 & 2\end{pmatrix}\mathbf{H}\}.$$

$\varepsilon H = \{\epsilon, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}\}$ $\quad \begin{pmatrix}1 & 2\end{pmatrix} H = \{\begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}, \begin{pmatrix}2 & 3\end{pmatrix}\}$

Define $\alpha\mathbf{H} * \beta\mathbf{H} \overset{\text{def}}{=} \alpha \circ \beta\mathbf{H}$**.**

i. **Find** $\varepsilon\mathbf{H} * \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{H}$**.**

$$\begin{aligned}\varepsilon H * \begin{pmatrix}1 & 3\end{pmatrix} H &\overset{def}{=} \left(\varepsilon \circ \begin{pmatrix}1 & 3\end{pmatrix}\right) H = \begin{pmatrix}1 & 3\end{pmatrix} H \\ &= \begin{pmatrix}1 & 2\end{pmatrix} H\end{aligned}$$

Thus $\varepsilon H * \begin{pmatrix}1 & 3\end{pmatrix} H = \begin{pmatrix}1 & 2\end{pmatrix} H.$

Here, the coset εH operated with the coset $\begin{pmatrix}1 & 2\end{pmatrix} H$ –using $\begin{pmatrix}1 & 3\end{pmatrix}$ as the coset representative rather than $\begin{pmatrix}1 & 2\end{pmatrix}$, results in the coset $\begin{pmatrix}1 & 2\end{pmatrix} H$.

Solutions to Problem 3 from Monday:

$\varepsilon H = \{\epsilon, (1\ \ 2\ \ 3), (1\ \ 3\ \ 2)\}$ $\qquad (1\ \ 2)\, H = \{(1\ \ 2), (1\ \ 3), (2\ \ 3)\}$

3.(a)ii. **Find $(1\ \ 2\ \ 3)\,\mathbf{H} * (1\ \ 2)\,\mathbf{H}$. Which of the two cosets do you get?**

$$\begin{aligned} (1\ \ 2\ \ 3)\, H * (1\ \ 2)\, H &\overset{def}{=} ((1\ \ 2\ \ 3) \circ (1\ \ 2))H \\ &= (1\ \ 3)\, H \end{aligned}$$

Because $(1\ \ 3)\, H = (1\ \ 2)\, H$, we found have found that once again, the coset εH – this time using $(1\ \ 2\ \ 3)$ as its representative –operated with the coset $(1\ \ 2)\, H$ is $(1\ \ 2)\, H$.

Solutions to Problem 3 from Monday:

$\varepsilon H = \{\epsilon, (1\ \ 2\ \ 3), (1\ \ 3\ \ 2)\}$ $\quad$ $(1\ \ 2)H = \{(1\ \ 2), (1\ \ 3), (2\ \ 3)\}$

3.(a) iii. **Find $(1\ \ 3\ \ 2)\,\mathbf{H} * (2\ \ 3)\,\mathbf{H}$. Which of the two cosets do you get?**

$$\begin{aligned}(1\ \ 3\ \ 2)H * (2\ \ 3)H &\overset{def}{=} ((1\ \ 3\ \ 2)\circ(2\ \ 3))H \\ &= (1\ \ 3)H\end{aligned}$$

And again, because $(1\ \ 3)H = (1\ \ 2)H$, we've found that a version of the coset εH combined with a version of the coset $(1\ \ 2)H$ gives the coset $(1\ \ 2)H$.

Solutions to Problem 3 from Monday:

3.(a) iv. **Compare your results to the first three questions. What happened? Was it what you expected, or something different?**

In each case, we had $\varepsilon H * \begin{pmatrix} 1 & 2 \end{pmatrix} H$, but written different ways.

Each time, I found that the result was the coset $\begin{pmatrix} 1 & 2 \end{pmatrix} H$, although it sometimes showed up under an alias.

Solutions to Problem 3 from Monday:

3.a. v. **In general, for this subgroup H, show that if $\alpha_1 \mathbf{H} = \alpha_2 \mathbf{H}$ and $\beta_1 \mathbf{H} = \beta_2 \mathbf{H}$, then $\alpha_1 \mathbf{H} * \beta_1 \mathbf{H} = \alpha_2 \mathbf{H} * \beta_2 \mathbf{H}$. (This shows that the operation $*$ is well-defined).**

Obviously, we have not shown that every possible combination for these two cosets gives the same result, let alone other pairs operated on each other.

However, we will skip showing that this operation is well-defined for now.

Solutions to Problem 3 from Monday:

3.(b) **Let K** $=<\begin{pmatrix}2 & 3\end{pmatrix}>$ **in** $\mathbf{S_3}$**, and consider the set of left cosets of** K

$$\mathbf{S_K} = \{\mathbf{K}, \begin{pmatrix}1 & 2\end{pmatrix}\mathbf{K}, \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{K}\}.$$

$K = \{\varepsilon, \boxed{\mathbf{(2\ 3)}}\}$, $(1\ 2)K = \{(1\ 2), (1\ 2\ 3)\}$, $(1\ 3)K = \{(1\ 3), (1\ 3\ 2)\}$

Define $\alpha\mathbf{K} * \beta\mathbf{K} \overset{\textbf{def}}{=} (\alpha \circ \beta)\mathbf{K}$**.**

$$\begin{aligned} \begin{pmatrix}1 & 2\end{pmatrix}K * \begin{pmatrix}1 & 3\end{pmatrix}K &\overset{def}{=} (\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix})K \\ &= \begin{pmatrix}1 & 3 & 2\end{pmatrix}K \\ \begin{pmatrix}1 & 2 & 3\end{pmatrix}K * \begin{pmatrix}1 & 3 & 2\end{pmatrix}K &\overset{def}{=} (\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 3 & 2\end{pmatrix})K \\ &= K \end{aligned}$$

Since $\begin{pmatrix}1 & 3 & 2\end{pmatrix}K \neq K$, these two different ways of doing the same operation on the same elements gives different results. Thus with this choice of subgroup, the operation $*$ is *not* well-defined!

Results:

$H \leq G$	$\{aH \mid a \in G\}$ group?	$H \lhd G$?
$n\mathbb{Z} \leq \mathbb{Z}$	yes	yes
$\langle\, (1 \quad 2)\,\rangle \leq S_3$	no	no
$\langle\, (2 \quad 3)\,\rangle \leq S_3$	no	no
$\langle\, (1 \quad 2 \quad 3)\,\rangle \leq S_3$	maybe	yes

In Class Work

Let a be an element of order 24, and consider $G =< a >= \{e, a, a^2, a^3, \ldots, a^{22}, a^{23}\}$. Clearly, $H =< a^6 >$ is a subgroup of G.

1. How do you know G/H is a group?
2. What is $|G/H|$?
3. List the elements of G/H.
4. What is another way of writing $a^{14}H$?
5. What is $a^3H * a^5H$?
6. What is the order of a^2H? (Here, I mean the order of the *element* a^2H in the group G/H, not the order of the *set* a^2H.)

Solutions:

Let a be an element of order 24, and consider $G =< a >= \{e, a, a^2, a^3, \ldots, a^{22}, a^{23}\}$. Clearly, $H =< a^6 >$ is a subgroup of G.

1. How do you know G/H is a group?

   Because G is cyclic, it is Abelian. All subgroups of Abelian groups are normal, and therefore H is normal.

   Thus, it follows from Theorem 9.2 that G/H is a group, under the operation $a^n H * a^m H = (a^n a^m)H = a^{n+m}H$.

Note: Because $|a| = 24$, we can rewrite this as

$$a^n H * a^m H = a^{n+m \mod 24} H.$$

Solutions:

2. What is $|G/H|$?

   $G/H =$ the set of all (left) cosets of H in G. Thus $|G/H|$ is the number of all (left) cosets of H in G, or in other words,

   $$|G/H| = [G : H] = \text{ (by Lagrange's Theorem) } \frac{|G|}{|H|}.$$

   Since $|H| = \frac{24}{6} = 4$, we have that $|G/H| = \frac{24}{4} = 6$.

Solutions:

3. List the elements of G/H.

$$G/H = \{eH, aH, a^2H, a^3H, a^4H, a^5H\}.$$

Solutions:

4. What is another way of writing $a^{14}H$?

   All elements of G have the form a^k for some k.

   From the lemma back in Chapter 7, we know that

$$\begin{aligned} a^{14}H = a^kH &\iff a^{-k}a^{14} \in H \\ &\iff a^{14-k \mod 24} \in H \\ &\iff a^{14-k \mod 24} \in \{a^6, a^{12}, a^{18}, e\} \\ &\iff 14-k \mod 24 \in \{6, 12, 18, 0\} \\ &\iff k \in \{8, 2, -4, 14\}. \end{aligned}$$

   Thus one other way of writing $a^{14}H$ is a^2H.

Solutions:

5. What is $a^3H * a^5H$?

   $a^3H * a^5H = (a^3a^5)H = a^8H = a^2H.$

Solutions:

6. What is the order of a^2H?

$$
\begin{aligned}
(a^2H)^k = eH &\iff a^{2k}H = eH \\
&\iff a^{2k} \in H \\
&\iff 2k \in \{0, 6, 12, 18\}.
\end{aligned}
$$

Since we want the smallest positive such k, $|a^2H| = 3$.