# In Class Work

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

1. List the elements of $H$, and discuss their order.
2. List the elements of $K$, and discuss their order.
3. Show $H \approx K$.
4. List the elements of $G/H$, and discuss their order.
5. List the elements of $G/K$, and discuss their order.
6. Show $G/H \not\approx G/K$.

# In Class Work

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

1. List the elements of $H$, and discuss their order.

   **Hint:** Be sure to first identify the operation and identity of $G$.

2. List the elements of $K$, and discuss their order.

3. Show $H \approx K$.

4. List the elements of $G/H$, and discuss their order.

5. List the elements of $G/K$, and discuss their order.

6. Show $G/H \not\approx G/K$.

# In Class Work

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

1. List the elements of $H$, and discuss their order.

   **Hint:** Be sure to first identify the operation and identity of $G$.

2. List the elements of $K$, and discuss their order.

3. Show $H \approx K$.

   **Hint:** Ponder what you know about these 2 subgroups, **or** construct an element-by-element map; check it's an isomorphism.

4. List the elements of $G/H$, and discuss their order.

5. List the elements of $G/K$, and discuss their order.

6. Show $G/H \not\approx G/K$.

# In Class Work

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

1. List the elements of $H$, and discuss their order.

   **Hint:** Be sure to first identify the operation and identity of $G$.

2. List the elements of $K$, and discuss their order.

3. Show $H \approx K$.

   **Hint:** Ponder what you know about these 2 subgroups, **or** construct an element-by-element map; check it's an isomorphism.

4. List the elements of $G/H$, and discuss their order.

   **Hint 1:** First list the elements of $G$.

5. List the elements of $G/K$, and discuss their order.

6. Show $G/H \not\approx G/K$.

# In Class Work

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H = < (2,3) >$, and $K = < (2,1) >$.

1. List the elements of $H$, and discuss their order.

   **Hint:** Be sure to first identify the operation and identity of $G$.

2. List the elements of $K$, and discuss their order.

3. Show $H \approx K$.

   **Hint:** Ponder what you know about these 2 subgroups, **or** construct an element-by-element map; check it's an isomorphism.

4. List the elements of $G/H$, and discuss their order.

   **Hint 1:** First list the elements of $G$.

   **Hint 2:** From context, do I mean order of an element or order of set?

5. List the elements of $G/K$, and discuss their order.

6. Show $G/H \not\approx G/K$.

# In Class Work

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

1. List the elements of $H$, and discuss their order.

   **Hint:** Be sure to first identify the operation and identity of $G$.

2. List the elements of $K$, and discuss their order.

3. Show $H \approx K$.

   **Hint:** Ponder what you know about these 2 subgroups, **or** construct an element-by-element map; check it's an isomorphism.

4. List the elements of $G/H$, and discuss their order.

   **Hint 1:** First list the elements of $G$.

   **Hint 2:** From context, do I mean order of an element or order of set?

   **Hint 3:** The op on $G/H$ is a natural carry-over from the op of $G$.

5. List the elements of $G/K$, and discuss their order.

6. Show $G/H \not\approx G/K$.

# Solutions:

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

1. List the elements of $H$, and discuss their order.

$$\begin{aligned} H &= \{(2,3),(0,1)\} \\ |(0,1)| &= 1 \\ |(2,3)| &= 2 \end{aligned}$$

2. List the elements of $K$, and discuss their order.

$$\begin{aligned} K &= \{(2,1),(0,1)\} \\ |(0,1)| &= 1 \\ |(2,1)| &= 2 \end{aligned}$$

# Solutions:

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

3. Show $H \approx K$.

   Since both $H$ and $K$ are cyclic groups of order 2, they're both isomorphic to $\mathbb{Z}_2$, and so they're isomorphic to each other.

   **OR:**

   Let $f : H \to K$ by $f((2,3)) = (2,1)$ and $f((0,1) = (0,1)$. This is clearly well-defined, 1-1, and onto, and it's operation preserving since $f((2,3)^2) = f(0,1) = (0,1) = (2,1)^2$.

# Solutions:

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

4. List the elements of $G/H$, and discuss their order.

$$G = \{(0,1),(0,3),(1,1),(1,3),(2,1),(2,3),(3,1),(3,3)\}$$

$$\begin{aligned}
(0,1)H = H &= (2,3)H \\
(0,3)H = \{(0,3),(2,1)\} &= (2,1)H \\
(1,1)H = \{(1,1),(3,3)\} &= (3,3)H \\
(1,3)H = \{(1,3),(3,1)\} &= (3,1)H
\end{aligned}$$

In $G/H$,

$$\begin{aligned}
& & |(0,1)H| = 1 \\
\left((0,3)H\right)^2 = (0,3)^2H = (0,1)H = eH &\Rightarrow & |(0,3)H| = 2 \\
\left((1,1)H\right)^2 = (1,1)^2H = (2,1)H = (0,3)H &\Rightarrow & |(1,1)H| = 4 \\
\left((1,3)H\right)^2 = (1,3)^2H = (2,1)H = (0,3)H &\Rightarrow & |(1,3)H| = 4
\end{aligned}$$

# Solutions:

$G/H = \{(0,1)H, (0,3)H, (1,1)H, (1,3)H\}$, and

$$|(0,1)H| = 1 \qquad |(0,3)H| = 2 \qquad |(1,1)H| = 4 \qquad |(1,3)H| = 4$$

Thus $G/H$ is a cyclic group of order 4 and so $G/H \approx \mathbb{Z}_4$

## Solutions:

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H = <(2,3)>$, and $K = <(2,1)>$.

5. List the elements of $G/K$, and discuss their order.

$$G = \{(0,1),(0,3),(1,1),(1,3),(2,1),(2,3),(3,1),(3,3)\}$$

$$\begin{aligned} (0,1)K = K &= (2,1)K \\ (0,3)K = \{(0,3),(2,3)\} &= (2,3)K \\ (1,1)K = \{(1,1),(3,1)\} &= (3,1)K \\ (1,3)K = \{(1,3),(3,3)\} &= (3,3)K \end{aligned}$$

In $G/K$,

$$\begin{aligned} & & |(0,1)K| = 1 \\ \big((0,3)K\big)^2 = (0,3)^2K = (0,1)K &\Rightarrow & |(0,3)K| = 2 \\ \big((1,1)K\big)^2 = (1,1)^2K = (2,1)K = (0,1)K &\Rightarrow & |(1,1)K| = 2 \\ \big((1,3)K\big)^2 = (1,3)^2K = (2,1)K = (0,1)K &\Rightarrow & |(1,3)K| = 2 \end{aligned}$$

# Solutions:

$G/K = \{(0,1)K, (0,3)K, (1,1)K, (1,3)K\}$, and

$$|(0,1)K| = 1 \qquad |(0,3)K| = 2 \qquad |(1,1)K| = 2 \qquad |(1,3)K| = 2$$

Thus $G/H$ is a non-cyclic group of order 4, and so $G/H \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$

# Solutions:

Let $G = \mathbb{Z}_4 \oplus U(4)$, $H =< (2,3) >$, and $K =< (2,1) >$.

6. Show $G/H \not\approx G/K$.

   $G/H$ is a cyclic group of order 4 while $G/K$ is not. Thus they can not be isomorphic.

   Even though $H$ and $K$ **are** isomorphic, $G/H$ and $G/K$ are **not**!

# Visualizing Factor Groups: $G = D_4$ and $K = \{R_0, R_{180}\}$

Cayley Table for $G$

| | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ | H | V | D | D' |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ | H | V | D | D' |
| $R_{180}$ | $R_{180}$ | $R_0$ | $R_{270}$ | $R_{90}$ | V | H | D' | D |
| $R_{90}$ | $R_{90}$ | $R_{270}$ | $R_{180}$ | $R_0$ | D' | D | H | V |
| $R_{270}$ | $R_{270}$ | $R_{90}$ | $R_0$ | $R_{180}$ | D | D' | V | H |
| H | H | V | D | D' | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ |
| V | V | H | D' | D | $R_{180}$ | $R_0$ | $R_{270}$ | $R_{90}$ |
| D | D | D' | V | H | $R_{270}$ | $R_{90}$ | $R_0$ | $R_{180}$ |
| D' | D' | D | H | V | $R_{90}$ | $R_{270}$ | $R_{180}$ | $R_0$ |

$K=\{R_0,R_{180}\}$ | $R_{90}K=\{R_{90}, R_{270}\}$ | $HK=\{H,V\}$ | $DK=\{D,D'\}$

# Visualizing Factor Groups: $G = D_4$ and $K = \{R_0, R_{180}\}$

Cayley Table for $G$
with the cosets of $K$ shaded

| | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ | H | V | D | D' |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ | H | V | D | D' |
| $R_{180}$ | $R_{180}$ | $R_0$ | $R_{270}$ | $R_{90}$ | V | H | D' | D |
| $R_{90}$ | $R_{90}$ | $R_{270}$ | $R_{180}$ | $R_0$ | D' | D | H | V |
| $R_{270}$ | $R_{270}$ | $R_{90}$ | $R_0$ | $R_{180}$ | D | D' | V | H |
| H | H | V | D | D' | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ |
| V | V | H | D' | D | $R_{180}$ | $R_0$ | $R_{270}$ | $R_{90}$ |
| D | D | D' | V | H | $R_{270}$ | $R_{90}$ | $R_0$ | $R_{180}$ |
| D' | D' | D | H | V | $R_{90}$ | $R_{270}$ | $R_{180}$ | $R_0$ |

| | | | |
|---|---|---|---|
| $K=\{R_0,R_{180}\}$ | $R_{90}K=\{R_{90}, R_{270}\}$ | $HK=\{H,V\}$ | $DK=\{D,D'\}$ |

# Visualizing Factor Groups: $G = D_4$ and $K = \{R_0, R_{180}\}$

Cayley Table for $G$
with the cosets of $K$ shaded

| | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ | H | V | D | D' |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ | H | V | D | D' |
| $R_{180}$ | $R_{180}$ | $R_0$ | $R_{270}$ | $R_{90}$ | V | H | D' | D |
| $R_{90}$ | $R_{90}$ | $R_{270}$ | $R_{180}$ | $R_0$ | D' | D | H | V |
| $R_{270}$ | $R_{270}$ | $R_{90}$ | $R_0$ | $R_{180}$ | D | D' | V | H |
| H | H | V | D | D' | $R_0$ | $R_{180}$ | $R_{90}$ | $R_{270}$ |
| V | V | H | D' | D | $R_{180}$ | $R_0$ | $R_{270}$ | $R_{90}$ |
| D | D | D' | V | H | $R_{270}$ | $R_{90}$ | $R_0$ | $R_{180}$ |
| D' | D' | D | H | V | $R_{90}$ | $R_{270}$ | $R_{180}$ | $R_0$ |

Cayley Table for $G/K$

| | K | $R_{90}$K | HK | DK |
|---|---|---|---|---|
| K | K | $R_{90}$K | HK | DK |
| $R_{90}$K | $R_{90}$K | K | DK | HK |
| HK | HK | DK | K | $R_{90}$K |
| DK | DK | HK | $R_{90}$K | K |

| | | | |
|---|---|---|---|
| K={$R_0$,$R_{180}$} | $R_{90}$K={$R_{90}$, $R_{270}$} | HK={H,V} | DK={D,D'} |