The Alternating Group A_4 of Even Permutations of $\{1,2,3,4\}$

In this table, the permutations of A_4 are designated as ε, α_2, . . . , α_{12}.

$\circ$	ε	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$(1) = \varepsilon$	ε	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$(12)(34) = \alpha_2$	α_2	ε	α_4	α_3	α_6	α_5	α_8	α_7	α_{10}	α_9	α_{12}	α_{11}
$(13)(24) = \alpha_3$	α_3	α_4	ε	α_2	α_7	α_8	α_5	α_6	α_{11}	α_{12}	α_9	α_{10}
$(14)(23) = \alpha_4$	α_4	α_3	α_2	ε	α_8	α_7	α_6	α_5	α_{12}	α_{11}	α_{10}	α_9
$(123) = \alpha_5$	α_5	α_8	α_6	α_7	α_9	α_{12}	α_{10}	α_{11}	ε	α_4	α_2	α_3
$(243) = \alpha_6$	α_6	α_7	α_5	α_8	α_{10}	α_{11}	α_9	α_{12}	α_2	α_3	ε	α_4
$(142) = \alpha_7$	α_7	α_6	α_8	α_5	α_{11}	α_{10}	α_{12}	α_9	α_3	α_2	α_4	ε
$(134) = \alpha_8$	α_8	α_5	α_7	α_6	α_{12}	α_9	α_{11}	α_{10}	α_4	ε	α_3	α_2
$(132) = \alpha_9$	α_9	α_{11}	α_{12}	α_{10}	ε	α_3	α_4	α_2	α_5	α_7	α_8	α_6
$(143) = \alpha_{10}$	α_{10}	α_{12}	α_{11}	α_9	α_2	α_4	α_3	ε	α_6	α_8	α_7	α_5
$(234) = \alpha_{11}$	α_{11}	α_9	α_{10}	α_{12}	α_3	ε	α_2	α_4	α_7	α_5	α_6	α_8
$(124) = \alpha_{12}$	α_{12}	α_{10}	α_9	α_{11}	α_4	α_2	ε	α_3	α_8	α_6	α_5	α_7

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H. (How do you know G/H is a group?)(You may assume that a factor group of an Abelian group is Abelian, which you will be showing for PS 8)

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$. Is the group cyclic?

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

   Helpful shorthand for cosets (just remember the elements are really sets!):

   $$\text{Write } a + 20\mathbb{Z} \text{ as } \bar{a}.$$

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

   Helpful shorthand for cosets (just remember the elements are really sets!):

   $$\text{Write } a + 20\mathbb{Z} \text{ as } \bar{a}.$$

   With this notation, the elements of G are the cosets

   $$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots, \overline{19}\}.$$

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

$$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots, \overline{19}\}.$$

By definition,

$$H = 4\mathbb{Z}/20\mathbb{Z} = \{\bar{a} \in \mathbb{Z}/20\mathbb{Z} | a \in 4\mathbb{Z}\}.$$

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

$$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots, \overline{19}\}.$$

By definition,

$$H = 4\mathbb{Z}/20\mathbb{Z} = \{\bar{a} \in \mathbb{Z}/20\mathbb{Z} | a \in 4\mathbb{Z}\}.$$

Hence

$$H = 4\mathbb{Z}/20\mathbb{Z} = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\}.$$

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

   Now we know:

   $$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots, \overline{19}\} \qquad \text{and} \qquad H = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\}.$$

   Why is G/H a group?

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

   Now we know:

   $$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots, \overline{19}\} \qquad \text{and} \qquad H = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\}.$$

   Why is G/H a group?

   Using that a factor group of an Abelian group is Abelian, *every* subgroup of G is normal. In particular, $H \triangleleft G$, and so G/H is a group.

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

Now we know:

$$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots, \overline{19}\} \qquad \text{and} \qquad H = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\}.$$

List the elements of G/H.
Since $|G| = 20$ and $|H| = 5$, from Lagrange's Theorem, we know that $|G/H| = 4$, so we know how many distinct cosets we're looking for.

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

Now we know:

$$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots, \overline{19}\} \qquad \text{and} \qquad H = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\}.$$

The elements of G/H are:

$$\begin{aligned}
\bar{0} + H &= H = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\} \\
&= \bar{4} + H = \bar{8} + H = \overline{12} + H = \overline{16} + H \\
\bar{1} + H &= \{\bar{1}, \bar{5}, \bar{9}, \overline{13}, \overline{17}\} \\
&= \bar{5} + H = \bar{9} + H = \overline{13} + H = \overline{17} + H \\
\bar{2} + H &= \{\bar{2}, \bar{6}, \overline{10}, \overline{14}, \overline{18}\} \\
&= \bar{6} + H = \overline{10} + H = \overline{14} + H = \overline{18} + H \\
\bar{3} + H &= \{\bar{3}, \bar{7}, \overline{11}, \overline{15}, \overline{19}\} \\
&= \bar{7} + H = \overline{11} + H = \overline{15} + H = \overline{19} + H
\end{aligned}$$

1. Let $G = \mathbb{Z}/20\mathbb{Z}$ and $H = 4\mathbb{Z}/20\mathbb{Z}$. List the elements of G/H.

Now we know:

$$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \dots, \overline{19}\} \qquad \text{and} \qquad H = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\}.$$

The elements of G/H are:

$$\begin{aligned}
\bar{0} + H &= H = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}\} \\
&= \bar{4} + H = \bar{8} + H = \overline{12} + H = \overline{16} + H \\
\bar{1} + H &= \{\bar{1}, \bar{5}, \bar{9}, \overline{13}, \overline{17}\} \\
&= \bar{5} + H = \bar{9} + H = \overline{13} + H = \overline{17} + H \\
\bar{2} + H &= \{\bar{2}, \bar{6}, \overline{10}, \overline{14}, \overline{18}\} \\
&= \bar{6} + H = \overline{10} + H = \overline{14} + H = \overline{18} + H \\
\bar{3} + H &= \{\bar{3}, \bar{7}, \overline{11}, \overline{15}, \overline{19}\} \\
&= \bar{7} + H = \overline{11} + H = \overline{15} + H = \overline{19} + H
\end{aligned}$$

Note: Since $|G/H| = 4$, we know that G/H is either cyclic and isomorphic to $\mathbb{Z}_4$ or else it is **not** cyclic, in which case it is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$. Is the group cyclic?

   We have seen that $\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}_n$. So ....

   First Guess: $(\mathbb{Z} \oplus \mathbb{Z}) < (4, 2) > \approx \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$. Is the group cyclic?

   We have seen that $\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}_n$. So ....

   First Guess: $(\mathbb{Z} \oplus \mathbb{Z}) < (4, 2) > \approx \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

   In that case, the order would be 8. Is this true?

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?

We have seen that $\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}_n$. So ....

First Guess: $(\mathbb{Z} \oplus \mathbb{Z}) < (4,2) > \approx \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

In that case, the order would be 8. Is this true?

$$\begin{aligned} < (4,2) > &= \{(0,0), (\pm 4, \pm 2), (\pm 8, \pm 4), (\pm 12, \pm 6), \ldots\} \\ &= \{(4n, 2n) | n \in \mathbb{Z}\} \end{aligned}$$

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?

We have seen that $\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}_n$. So ....

First Guess: $(\mathbb{Z} \oplus \mathbb{Z}) < (4,2) > \approx \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

In that case, the order would be 8. Is this true?

$$\begin{aligned} < (4,2) > &= \{(0,0), (\pm 4, \pm 2), (\pm 8, \pm 4), (\pm 12, \pm 6), \ldots\} \\ &= \{(4n, 2n) | n \in \mathbb{Z}\} \end{aligned}$$

A few elts (all of the form $(n,0)+ < (4,2) >$) of $\mathbb{Z} \oplus \mathbb{Z}/ < (4,2) >$:

$(0,0)+ < (4,2) >, (1,0)+ < (4,2) >, (2,0)+ < (4,2) >,$
$(3,0)+ < (4,2) >, \ (4,0)+ < (4,2) > \ldots.$

These are all distinct.

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?

We have seen that $\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}_n$. So ....

First Guess: $(\mathbb{Z} \oplus \mathbb{Z}) < (4,2) > \approx \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

In that case, the order would be 8. Is this true?

$$\begin{aligned} < (4,2) > &= \{(0,0), (\pm 4, \pm 2), (\pm 8, \pm 4), (\pm 12, \pm 6), \dots\} \\ &= \{(4n, 2n) | n \in \mathbb{Z}\} \end{aligned}$$

A few elts (all of the form $(n,0) + < (4,2) >$) of $\mathbb{Z} \oplus \mathbb{Z}/ < (4,2) >$:

$(0,0) + < (4,2) >, (1,0) + < (4,2) >, (2,0) + < (4,2) >,$
$(3,0) + < (4,2) >$, $(4,0) + < (4,2) > \dots.$

These are all distinct.

In general, $(n,0) + < (4,2) > \neq (m,0) + < (4,2) >$ as long as $n \neq m$.

(To see this, remember that $aH = bH \iff a^{-1}b \in H$.)

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?

We have seen that $\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}_n$. So ....

First Guess: $(\mathbb{Z} \oplus \mathbb{Z}) < (4,2) > \approx \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

In that case, the order would be 8. Is this true?

$$\begin{aligned} < (4,2) > &= \{(0,0), (\pm 4, \pm 2), (\pm 8, \pm 4), (\pm 12, \pm 6), \ldots\} \\ &= \{(4n, 2n) | n \in \mathbb{Z}\} \end{aligned}$$

A few elts (all of the form $(n,0)+ < (4,2) >$) of $\mathbb{Z} \oplus \mathbb{Z}/ < (4,2) >$:

$(0,0)+ < (4,2) >, (1,0)+ < (4,2) >, (2,0)+ < (4,2) >,$
$(3,0)+ < (4,2) >, \ (4,0)+ < (4,2) > \ldots.$

These are all distinct.

In general, $(n,0)+ < (4,2) > \neq (m,0)+ < (4,2) >$ as long as $n \neq m$.

(To see this, remember that $aH = bH \iff a^{-1}b \in H$.)

Thus there are an infinite number of distinct cosets, and therefore $|\mathbb{Z} \oplus \mathbb{Z}/ < (4,2) > | = \infty$.

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$. Is the group cyclic?
Is $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$ cyclic?

Since the order of the group is infinite, if it is cyclic then every non-identity element must have infinite order.

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$. Is the group cyclic?
Is $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$ cyclic?

Since the order of the group is infinite, if it is cyclic then every non-identity element must have infinite order.

Goal: find $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ but not in $< (4, 2) >$ such that $(a, b)+ < (4, 2) >$ has finite order.

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?
Is $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$ cyclic?

Since the order of the group is infinite, if it is cyclic then every non-identity element must have infinite order.

Goal: find $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ but not in $< (4,2) >$ such that $(a, b)+ < (4,2) >$ has finite order.

The quickest order to find is order 2, so try to find an elt with $2\big[(a, b)+ < (4,2) >\big] = (0,0)+ < (4,2) >$

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$. Is the group cyclic?
Is $(\mathbb{Z} \oplus \mathbb{Z})/ < (4, 2) >$ cyclic?

Since the order of the group is infinite, if it is cyclic then every non-identity element must have infinite order.

Goal: find $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ but not in $< (4, 2) >$ such that

$$2((a, b)+ < (4, 2) >= (0, 0)+ < (4, 2) > .$$

Want: $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ but not in $< (4, 2) >$ such that $(2a, 2b) \in < (4, 2) >$.

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?
Is $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$ cyclic?

Since the order of the group is infinite, if it is cyclic then every non-identity element must have infinite order.

Goal: find $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ but not in $< (4,2) >$ such that

$$2((a,b) + < (4,2) >= (0,0) + < (4,2) > .$$

Want: $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ but not in $< (4,2) >$ such that $(2a, 2b) \in < (4,2) >$.

$(2,1)$ seems like it would work.

Check:

$$2((2,1) + < (4,2) >) = (4,2) + < (4,2) > = < 4, 2 > .$$

2. Determine the order of $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Is the group cyclic?
Is $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$ cyclic?

Since the order of the group is infinite, if it were cyclic, every non-identity element would have infinite order.

$$(2,1)+ < (4,2) > \neq (0,0)+ < (4,2) >$$

and

$$\left((2,1)+ < (4,2) >\right)^{"2"} = (0,0)+ < (4,2) >,$$

so we have found an element of order 2 in $(\mathbb{Z} \oplus \mathbb{Z})/ < (4,2) >$. Thus this group is **not** cyclic.