## Properties of Elements Under Homomorphisms — Those that we (essentially) already saw when studying isomorphisms

Let $\phi : G \to \bar{G}$ be a homomorphism, and let $g \in G$.

| Properties homomorphisms share with isomorphisms | Properties of homomorphisms that differ from those of isomorphisms |
|---|---|
| 1. $\phi(e_G) = e_{\bar{G}}$<br>2. $\phi(g^n) = (\phi(g))^n \; \forall \; n \in \mathbb{Z}$. | 3. If $\lvert g\rvert$ is finite, $\lvert\phi(g)\rvert$ divides $\lvert g\rvert$. |

**Note:** Just as with isomorphisms, from Property 2 we know that $\phi(g^{-1}) = \left(\phi(g)\right)^{-1}$.

## Properties of Subgroups Under Homomorphisms — Those that are shared with isomorphisms

Let $\phi : G \to \bar{G}$ be a homomorphism, and let $H$ be a subgroup of $G$.

1. $\phi(H) = \{\phi(h) | h \in H\}$ is a subgroup of $\bar{G}$.
2. If $H$ is cyclic, then $\phi(H)$ is cyclic.
3. If $H$ is Abelian, then $\phi(H)$ is Abelian.

**Note:** From Property 1, we know that $\phi(G)$ is a subgroup of $\bar{G}$.

# Properties of Homomorphisms

Let $\phi : G \to \bar{G}$ be a homomorphism, let $g \in G$, and let $H \leq G$.

| Properties of elements | Properties of subgroups |
|---|---|
| 1. $\phi(e_G) = e_{\bar{G}}$ | 1. $\phi(H) \leq \bar{G}$. |
| 2. $\phi(g^n) = (\phi(g))^n \forall n \in \mathbb{Z}$. | 2. $H$ cyclic $\Longrightarrow \phi(H)$ cyclic. |
| 3. If $\lvert g \rvert$ is finite, $\lvert \phi(g) \rvert \,\Big\vert\, \lvert g \rvert$. | 3. $H$ Abelian $\Longrightarrow \phi(H)$ Abelian. |
| | 7. $\bar{K} \leq \bar{G} \Longrightarrow \phi^{-1}(\bar{K}) \leq G$. |

## Properties of Homomorphisms

Let $\phi : G \to \bar{G}$ be a homomorphism, let $g \in G$, and let $H \leq G$.

| Properties of elements | Properties of subgroups |
|---|---|
| 1. $\phi(e_G) = e_{\bar{G}}$ | 1. $\phi(H) \leq \bar{G}$. |
| 2. $\phi(g^n) = (\phi(g))^n \forall n \in \mathbb{Z}$. | 2. $H$ cyclic $\implies \phi(H)$ cyclic. |
| 3. If $\lvert g \rvert$ is finite, $\lvert \phi(g) \rvert \, \Big\vert \, \lvert g \rvert$. | 3. $H$ Abelian $\implies \phi(H)$ Abelian. |
| | 7. $\bar{K} \leq \bar{G} \implies \phi^{-1}(\bar{K}) \leq G$. |
| | 4. $H \lhd G \implies \phi(H) \lhd \phi(G)$ |

# Properties of Homomorphisms

Let $\phi : G \to \bar{G}$ be a homomorphism, let $g \in G$, and let $H \leq G$.

| Properties of elements | Properties of subgroups |
|---|---|
| 1. $\phi(e_G) = e_{\bar{G}}$ | 1. $\phi(H) \leq \bar{G}$. |
| 2. $\phi(g^n) = (\phi(g))^n \forall n \in \mathbb{Z}$. | 2. $H$ cyclic $\Longrightarrow \phi(H)$ cyclic. |
| 3. If $\lvert g \rvert$ is finite, $\lvert\phi(g)\rvert \,\Big|\, \lvert g \rvert$. | 3. $H$ Abelian $\Longrightarrow \phi(H)$ Abelian. |
| | 7. $\bar{K} \leq \bar{G} \Longrightarrow \phi^{-1}(\bar{K}) \leq G$. |
| | 4. $H \triangleleft G \Longrightarrow \phi(H) \triangleleft \phi(G)$ |
| | 8. $\bar{K} \triangleleft \bar{G} \Longrightarrow \phi^{-1}(\bar{K}) \triangleleft G$. |

# In Class Work

1. Find the kernel of the homomorphism $p : G \oplus H \to G$ by $p(g, h) = g$.

2. Find the kernel of the homomorphism $i : H \to G \oplus H$ by $i(h) = (e_G, h)$.

# Solutions:

1. Find the kernel of the homomorphism $p : G \oplus H \to G$ by $p(g, h) = g$.

$$\begin{aligned} Ker(p) &= \{(g, h) \in G \oplus H | p(g, h) = e_G\} \\ &= \{(g, h) | g = e_G\} \\ &= \{(e_G, h) | h \in H\} \\ &= \{e_G\} \oplus H. \end{aligned}$$

# Solutions:

2. Find the kernel of the homomorphism $i : H \to G \oplus H$ by $i(h) = (e_G, h)$.

$$\begin{aligned} Ker(i) &= \{h \in H | i(h) \\ &= e_{G\oplus H} \\ &= (e_G, e_H)\} \\ &= \{h \in H | (e_G, h) \\ &= (e_G, e_H)\} \\ &= \{e_H\}. \end{aligned}$$

**Note:** We could have done this faster. Since $i$ is a homomorphism, we *know* the identity goes to the identity. Since it's 1-1, nothing besides the identity can map to the identity. Thus the kernel, which is the set of all things that map to the identity, contains only the identity.