Properties of Homomorphisms

Recall: A function $\phi : G \to \bar{G}$ is a homomorphism if

$$\phi(ab) = \phi(a)\phi(b) \forall a, b \in G.$$

Let $\phi : G \to \bar{G}$ be a homomorphism, let $g \in G$, and let $H \leq G$.

Properties of elements	Properties of subgroups
1. $\phi(e_G) = e_{\bar{G}}$	1. $\phi(H) \leq \bar{G}$.
2. $\phi(g^n) = (\phi(g))^n \forall n \in \mathbb{Z}$.	2. H cyclic $\implies \phi(H)$ cyclic.
3. If $\lvert g \rvert$ is finite, $\lvert \phi(g) \rvert \bigm\vert \lvert g \rvert$.	3. H Abelian $\implies \phi(H)$ Abelian.
	7. $\bar{K} \leq \bar{G} \implies \phi^{-1}(\bar{K}) \leq G$.
	4. $H \triangleleft G \implies \phi(H) \triangleleft \phi(G)$
	8. $\bar{K} \triangleleft \bar{G} \implies \phi^{-1}(\bar{K}) \triangleleft G$.

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3. If $\lvert g\rvert$ is finite, $\lvert\phi(g)\rvert$ divides $\lvert g\rvert$.	3. H Abelian $\Longrightarrow \phi(H)$ Abelian.
4. $Ker(\phi) \leq G$	4. $H \triangleleft G \Longrightarrow \phi(H) \triangleleft \phi(G)$
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Remember that

$$Ker(\phi) \overset{def}{=} \{g \in G \mid \phi(g) = id_{\bar{G}}\}.$$

In Class Work

1. Find the kernel of the homomorphism $p : G \oplus H \to G$ by $p(g, h) = g$.

2. Find the kernel of the homomorphism $i : H \to G \oplus H$ by $i(h) = (e_G, h)$.

3. Let G be a group of permutations. For each $\sigma \in G$, define

$$sgn(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

   Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel?

Solutions:

1. Find the kernel of the homomorphism $p : G \oplus H \to G$ by $p(g,h) = g$.

$$\begin{aligned} Ker(p) &\stackrel{def}{=} \{(g,h) \in G \oplus H \big| p(g,h) = e_G\} \\ &= \{(g,h) \big| g = e_G\} \\ &= \{(e_G,h) \big| h \in H\} \\ &= \{e_G\} \oplus H. \end{aligned}$$

Solutions:

2. Find the kernel of the homomorphism $i : H \to G \oplus H$ by $i(h) = (e_G, h)$.

$$
\begin{aligned}
Ker(i) &\overset{def}{=} \{h \in H \big| i(h) = e_{G\oplus H} = (e_G, e_H)\} \\
&= \{h \in H \big| (e_G, h) = (e_G, e_H)\} \\
&= \{e_H\}.
\end{aligned}
$$

OR...

Since i is a homomorphism, $i(e_H) = e_{G\oplus H}$.

Since we showed Wednesday that i is 1-1, nothing besides the identity can map to the identity. Thus the kernel, which is the set of all things that map to the identity, contains only the identity.

Notice: This shows that the kernel of *any* 1-1 homomorphism consists only of the identity!

Solutions:

3. Let G be a group of permutations. For each $\sigma \in G$, define

$$sgn(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

   Prove that *sgn* is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel?

To show that *sgn* is a homomorphism, NTS *sgn* is a well-defined function and is operation-preserving.

Solutions:

3. Let G be a group of permutations. For each $\sigma \in G$, define

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   Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel?

Is *sgn* well-defined?

Suppose that $\sigma_1 = \sigma_2$.

Then since every permutation's factorization into transpositions will be either always odd or always even, either both σ_1 and σ_2 are even *or* both σ_1 and σ_2 are odd.

Thus $sgn(\sigma_1) = sgn(\sigma_2)$, and so *sgn* is a well-defined function.

Solutions to 3, continued

Recall: $sgn(\sigma) \stackrel{def}{=} \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$

Is *sgn* operation-preserving?

$$sgn(\alpha\beta) \;=\; \begin{cases} +1 & \text{if } \alpha\beta \text{ is even} \\ -1 & \text{if } \alpha\beta \text{ is odd} \end{cases}$$

Solutions to 3, continued

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$\alpha\beta$ is even $\Leftrightarrow \alpha, \beta$ both even or both odd

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$$sgn(\alpha\beta) = \begin{cases} +1 & \text{if } \alpha,\ \beta \text{ both even or both odd} \\ -1 & \text{if one is even, the other odd} \end{cases}$$

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$$\begin{aligned} sgn(\alpha\beta) &= \begin{cases} +1 & \text{if } \alpha,\ \beta \text{ both even or both odd} \\ -1 & \text{if one is even, the other odd} \end{cases} \\ &= \begin{cases} +1 & \text{if } sgn(\alpha) = sgn(\beta) \\ -1 & \text{if } sgn(\alpha) \neq sgn(\beta) \end{cases} \end{aligned}$$

Solutions to 3, continued

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$$sgn(\alpha\beta) = \begin{cases} +1 & \text{if } \alpha\beta \text{ is even} \\ -1 & \text{if } \alpha\beta \text{ is odd} \end{cases} \qquad \begin{matrix} \alpha\beta \text{ is even} \Leftrightarrow \alpha, \beta \\ \text{both even or both odd} \end{matrix}$$

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Solutions to 3, continued

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Solutions to 3, continued

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$$\begin{aligned} &= \begin{cases} +1 & \text{if } sgn(\alpha)sgn(\beta) = +1 \\ -1 & \text{if } sgn(\alpha)sgn(\beta) = -1 \end{cases} \\ &= sgn(\alpha)sgn(\beta) \end{aligned}$$

Thus *sgn* preserves the group operation

Solutions to 3, continued

$$sgn(\sigma) \overset{def}{=} \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Since *sgn* is a well-defined function that preserves the group operation, *sgn* is indeed a homomorphism.

Solutions to 3, continued

Kernel of *sgn*?

$$sgn(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Recall: The kernel of a homomorphism is the set of all elements in the domain that map to the **identity of the range.**

The identity of the multiplicative group $\{-1, +1\}$ is 1.

Solutions to 3, continued

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Thus

$$\begin{aligned} Ker(sgn) &= \{\alpha \in G | sgn(\alpha) = 1\} \\ &= \{\alpha \in G | \alpha \text{ is even}\} \end{aligned}$$

Solutions to 3, continued

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If G happens to be one of the S_n, then $Ker(sgn) = A_n$.

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Peer Review Exchange

		Luke, gps of order 8	$\leftrightarrow$	Marie, gps of order 9
		Dan, Q_4	$\leftrightarrow$	Sam, $\mathbb{Z}[i]$
		Alfred, M	$\leftrightarrow$	Becky H
		Shawn, $U(13)$, $U(21)$	$\leftrightarrow$	Erin, $\mathbb{Q}_{2,3}$
		Christina, $\mathbb{Z}_2[x]$	$\leftrightarrow$	Stephanie, $GL(2, \mathbb{Z}_2)$
		Laura, $\mathbb{Z}_2[x]$	$\leftrightarrow$	Aubrie, $\mathbb{C}_{10}$
		Tiffany, $\mathcal{F}$	$\rightarrow$	Eric, $\mathcal{T}_2$
Abbe, D_6	$\rightarrow$	Rebecca, $\mathcal{T}_2$	$\rightarrow$	Roy, $\mathcal{F}$
$\nwarrow\leftarrow\leftarrow\leftarrow$	$\leftarrow$	$\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow$	$\leftarrow$	$\leftarrow\leftarrow\hookleftarrow$