Recall: Properties of Homomorphisms

Let $\phi : G \to \bar{G}$ be a homomorphism, let $g \in G$, and let $H \leq G$.

Properties of elements	Properties of subgroups
1. $\phi(e_G) = e_{\bar{G}}$	1. $\phi(H) \leq \bar{G}$.
2. $\phi(g^n) = (\phi(g))^n$ for all $n \in \mathbb{Z}$.	2. H cyclic $\Rightarrow \phi(H)$ cyclic.
3. If $\lvert g\rvert$ is finite, $\lvert\phi(g)\rvert$ divides $\lvert g\rvert$.	3. H Abelian $\Rightarrow \phi(H)$ Abelian.
4. $Ker(\phi) \leq G$	4. $H \triangleleft G \Rightarrow \phi(H) \triangleleft \phi(G)$
In fact, $Ker(\phi) \triangleleft G$	
5. $\phi(a) = \phi(b)$	
$\Leftrightarrow aKer(\phi) = bKer(\phi)$	
6. $\phi(g) = \bar{g} \Rightarrow \phi^{-1}(\bar{g}) = gKer(\phi)$	
	7. $\bar{K} \leq \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \leq G$.
	8. $\bar{K} \triangleleft \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \triangleleft G$.

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5. $\phi(a) = \phi(b)$ $\Leftrightarrow aKer(\phi) = bKer(\phi)$	5. $\lvert Ker(\phi)\rvert = n \Rightarrow \phi$ is an n-to-1 map
6. $\phi(g) = \bar{g} \Rightarrow \phi^{-1}(\bar{g}) = gKer(\phi)$	7. $\bar{K} \leq \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \leq G$. 8. $\bar{K} \lhd \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \lhd G$.
$Ker(\phi) = \{e_G\} \Leftrightarrow \phi$ is $1-1$	9. ϕ onto and $Ker(\phi) = \{e_G\} \Rightarrow \phi$ an isomorphism.

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6. $\phi(g) = \bar{g} \Rightarrow \phi^{-1}(\bar{g}) = gKer(\phi)$	6. $\lvert H\rvert = n \Rightarrow \lvert\phi(H)\rvert$ divides n 7. $\bar{K} \leq \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \leq G$. 8. $\bar{K} \triangleleft \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \triangleleft G$.
$Ker(\phi) = \{e_G\} \Leftrightarrow \phi$ is $1-1$	9. ϕ onto and $Ker(\phi) = \{e_G\} \Rightarrow \phi$ an isomorphism.

In Class Work

1. Prove that $(A \oplus B)/(A \oplus \{e\})$ is isomorphic to B, by identifying a homomorphism from $A \oplus B \to B$ that has $A \oplus \{e\}$ as its kernel.
2. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and that $Ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$, determine *all* elements that map to 9.

Solutions:

1. Prove that $(A \oplus B)/(A \oplus \{e\})$ is isomorphic to B.

   Define $\phi : A \oplus B \to B$ by $\phi((a,b)) = b$.

   From previous in-class work, we already know that ϕ is a well-defined *onto* homomorphism. In fact, we know (more or less) from Friday what the kernel is.

   $$Ker(\phi) = A \oplus \{e\}.$$

   The easiest way to do this problem is to use the 1st isomorphism theorem.

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   The easiest way to do this problem is to use the 1st isomorphism theorem.

   The 1st Isomorphism Thm tells us

   $A \oplus B/Ker(\phi) \approx \phi(A \oplus B)$, or $A \oplus B/Ker(\phi) \approx B$, since ϕ is onto.

   Thus, since $Ker(\phi) = A \oplus \{e\}$, $A \oplus B/A \oplus \{e\} \approx B$.

Solutions:

2. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and that $Ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$, determine *all* elements that map to 9.
    - We need to find $\phi^{-1}(9)$.

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    - Thm 10.1 Property 5 states:

$$\text{if } \phi(g) = \bar{g}, \text{ then } \phi^{-1}(\bar{g}) = gKer(\phi).$$

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    - In $\mathbb{Z}_{30}$, the operation is addition mod 30.
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   so the elements that map to 9 are 3, 13, and 23.