If $\phi : G \to \bar{G}$ is a homomorph, $g \in G$, and $H \leq G$:

Properties of elements	Properties of subgroups
1. $\phi(e_G) = e_{\bar{G}}$	1. $\phi(H) \leq \bar{G}$.
2. $\phi(g^n) = (\phi(g))^n$ for all $n \in \mathbb{Z}$.	2. H cyclic $\Rightarrow \phi(H)$ cyclic.
3. If $\lvert g \rvert$ is finite, $\lvert \phi(g) \rvert$ divides $\lvert g \rvert$.	3. H Abelian $\Rightarrow \phi(H)$ Abelian.
4. $Ker(\phi) \leq G$ In fact, $Ker(\phi) \triangleleft G$	4. $H \triangleleft G \Rightarrow \phi(H) \triangleleft \phi(G)$
5. $\phi(a) = \phi(b)$ $\Leftrightarrow aKer(\phi) = bKer(\phi)$	5. $\lvert Ker(\phi) \rvert = n \Rightarrow \phi$ is an n-to-1 map
6. $\phi(g) = \bar{g} \Rightarrow \phi^{-1}(\bar{g}) = gKer(\phi)$	6. $\lvert H \rvert = n \Rightarrow \lvert \phi(H) \rvert$ divides n
	7. $\bar{K} \leq \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \leq G$.
	8. $\bar{K} \triangleleft \bar{G} \Rightarrow \phi^{-1}(\bar{K}) \triangleleft G$.
$Ker(\phi) = \{e_G\} \Leftrightarrow \phi$ is $1-1$	9. ϕ onto and $Ker(\phi) = \{e_G\} \Rightarrow \phi$ an isomorphism.

1st Isomorphism Thm: Let $\phi : G \to \bar{G}$ be a group homomorphism. Then the mapping $\bar{\phi} : G/Ker(\phi) \to \phi(G)$, defined by $\bar{\phi}(gKer(\phi)) = \phi(g)$, is an isomorphism. In other words, $G/Ker(\phi) \approx \phi(G)$.

In Class Work

1. Prove that $(A \oplus B)/(A \oplus \{e\})$ is isomorphic to B, by identifying a homomorphism from $A \oplus B \to B$ that has $A \oplus \{e\}$ as its kernel.
2. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and that $Ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$, determine *all* elements that map to 9.
3. Suppose that there is a homomorphism ϕ from $\mathbb{Z}_{17}$ to some group, and that ϕ is not one-to-one. Determine ϕ.
4. If ϕ is a homomorphism from $\mathbb{Z}_{30}$ onto a group of order 5, determine the kernel of ϕ.
5. How many homomorphisms are there from $\mathbb{Z}_{20}$ *onto* $\mathbb{Z}_{10}$? How many are there to $\mathbb{Z}_{10}$? (That is, how many are there that may or may not be onto?)

Solutions:

1. Prove that $(A \oplus B)/(A \oplus \{e\})$ is isomorphic to B.

   Define $\phi : A \oplus B \to B$ by $\phi((a,b)) = b$.

   From previous in-class work, we already know that ϕ is a well-defined *onto* homomorphism. In fact, we know (more or less) from Friday before break what the kernel is.

   $$Ker(\phi) = A \oplus \{e\}.$$

   The easiest way to do this problem is to use the 1st isomorphism theorem.

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   The easiest way to do this problem is to use the 1st isomorphism theorem.

   The 1st Isomorphism Thm tells us

   $A \oplus B/Ker(\phi) \approx \phi(A \oplus B)$, or $A \oplus B/Ker(\phi) \approx B$, since ϕ is onto.

   Thus, since $Ker(\phi) = A \oplus \{e\}$, $A \oplus B/A \oplus \{e\} \approx B$.

Solutions:

2. Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and that $Ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$, determine *all* elements that map to 9.

   - We need to find $\phi^{-1}(9)$.

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   so the elements that map to 9 are 3, 13, and 23.

3. Suppose that there is a homomorphism ϕ from $\mathbb{Z}_{17}$ to some group, and that ϕ is not one-to-one. Determine ϕ.

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- Thus every element in $\mathbb{Z}_{17}$ maps to the identity, and so $\phi : \mathbb{Z}_{17} \to G$ must be $\phi(n) = e$ for all $n \in \mathbb{Z}_{17}$.

4. If ϕ is a homomorphism from $\mathbb{Z}_{30}$ onto a group of order 5, determine the kernel of ϕ.

   - Let $\phi : \mathbb{Z}_{30} \stackrel{onto}{\rightarrow} G$, where G is a group of order 5. Then by the 1st isomorphism theorem,

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- Because $\mathbb{Z}_{30}$ is cyclic (of order 30), it has exactly one subgroup of order 6, so there's only one possibility for $Ker(\phi)$:

$$Ker(\phi) = < 5 > = \{0, 5, 10, 15, 20, 25\}.$$

5. How many homomorphisms are there from $\mathbb{Z}_{20}$ *onto* $\mathbb{Z}_{10}$? How many are there to $\mathbb{Z}_{10}$? (That is, how many are there that may or may not be onto?)

Onto:

- Let $\phi : \mathbb{Z}_{20} \overset{onto}{\rightarrow} \mathbb{Z}_{10}$ be a homomorphism.

  $\mathbb{Z}_{20} = <1>$, so for all $n \in \mathbb{Z}_{20}$,

  $\phi(n) = \phi(1 + \ldots + 1 \bmod 20) = \phi(1) + \ldots \phi(1) \bmod 10 = n\phi(1) \bmod 10.$

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- Thus we must have $\phi(1) \in U(10) = \{1, 3, 7, 9\}$, so there are exactly 4 homomorphisms from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_{10}$:

$\phi(n) = n \quad \phi(n) = 3n \text{ mod } 10 \quad \phi(n) = 7n \text{ mod } 10 \quad \phi(n) = 9n \text{ mod } 10$

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- On the other hand, 1 can go anywhere in $\mathbb{Z}_{10}$: we'll always end up with 0 mapping to 0, because $0 = 20 \cdot 1 \mod 20$ in $\mathbb{Z}_{20}$, so $\phi(0) = 20\phi(1) \mod 10 = 0$. Thus there's *at least* 1 homomorphism per destination.

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- Thus there are 10 possible homomorphisms from $\mathbb{Z}_{20}$ *into* $\mathbb{Z}_{10}$.