Recall:

- **Definition:** A subgroup H of a group G is a **normal subgroup** iff $aH = Ha$ for all $a \in G$. We denote a normal subgroup by $H \triangleleft G$.

Recall:

- **Definition:** A subgroup H of a group G is a **normal subgroup** iff $aH = Ha$ for all $a \in G$. We denote a normal subgroup by $H \triangleleft G$.

- It is possible for a subgroup H to be normal even if $ah \neq ha$ for all $h \in H$ and for all $a \in G$.

Recall:

- **Definition:** A subgroup H of a group G is a **normal subgroup** iff $aH = Ha$ for all $a \in G$. We denote a normal subgroup by $H \triangleleft G$.
- It is possible for a subgroup H to be normal even if $ah \neq ha$ for all $h \in H$ and for all $a \in G$.
- **Claim 1:** If G is Abelian, then *every* subgroup H of G is normal.
- **Claim 2:** Let G be any group. Then $Z(G)$ is normal in G.
- **Claim 3:** If $|G : H| = 2$, then $H \triangleleft G$.

  Remember: Having index 2 means H has only two left cosets in G.
- **Claim 4:** (from exam) If G is a finite group and H is the only subgroup of order n, then $H \triangleleft G$.

Hints on Worksheet

1. Whether you decide to use the definition of normality or the Normal Subgroup Test, just dive in and calculate however many of the cosets you need to, to draw your conclusion.

Hints on Worksheet

1. Whether you decide to use the definition of normality or the Normal Subgroup Test, just dive in and calculate however many of the cosets you need to, to draw your conclusion.
2. What is $|S_n|$? What is $|A_n|$?

Hints on Worksheet

1. Whether you decide to use the definition of normality or the Normal Subgroup Test, just dive in and calculate however many of the cosets you need to, to draw your conclusion.
2. What is $|S_n|$? What is $|A_n|$?
2. How many cosets of A_n are there? That is, what is $[S_n : A_n]$?

Hints on Worksheet

1. Whether you decide to use the definition of normality or the Normal Subgroup Test, just dive in and calculate however many of the cosets you need to, to draw your conclusion.
2. What is $|S_n|$? What is $|A_n|$?
2. How many cosets of A_n are there? That is, what is $[S_n : A_n]$?
3. (a)(v)

$$\begin{aligned} \alpha_1 H = \alpha_2 H &\quad\Rightarrow\quad \text{There exist } h_1, h_2 \in H \text{ such that } \alpha_1 h_1 = \alpha_2 h_2 \\ \beta_1 H = \beta_2 H &\quad\Rightarrow\quad \text{There exist } g_1, g_2 \in H \text{ such that } \beta_1 g_1 = \beta_2 g_2 \end{aligned}$$

Hints on Worksheet

1. Whether you decide to use the definition of normality or the Normal Subgroup Test, just dive in and calculate however many of the cosets you need to, to draw your conclusion.
2. What is $|S_n|$? What is $|A_n|$?
2. How many cosets of A_n are there? That is, what is $[S_n : A_n]$?
3. (a)(v)

$$\begin{aligned} \alpha_1 H = \alpha_2 H \quad &\Rightarrow \quad \text{There exist } h_1, h_2 \in H \text{ such that } \alpha_1 h_1 = \alpha_2 h_2 \\ \beta_1 H = \beta_2 H \quad &\Rightarrow \quad \text{There exist } g_1, g_2 \in H \text{ such that } \beta_1 g_1 = \beta_2 g_2 \end{aligned}$$

3 (a)(v)Use these to rewrite α_1, β_1 in terms of α_2, β_2, and work with $\alpha_1 H * \beta_1 H$.

Solutions:

1. **Let $\mathbf{K} = \{\varepsilon, (2\ \ 3)\}$. Is $\mathbf{K} \lhd \mathbf{S_3}$?**

Method 1: Use the definition of normality – look at αK and $K\alpha$ for all $\alpha \in S_3$.

$S_3 = \{\varepsilon, (1\ \ 2), (1\ \ 3), (2\ \ 3), (1\ \ 2\ \ 3), (1\ \ 3\ \ 2)\}$

$\alpha K = K = K\alpha\ \forall\ \alpha \in K \Rightarrow \varepsilon K = K\varepsilon$ and $(2\ \ 3)\,K = K\,(2\ \ 3)$.

It remains to check those $\alpha \notin K$:

α	αK	$K\alpha$
$(1\ \ 2)$	$\{(1\ \ 2), (1\ \ 2\ \ 3)\}$	$\{(1\ \ 2), (1\ \ 3\ \ 2)\}$

Since there exists an α for which $\alpha K \neq K\alpha$, $K = \{\varepsilon, (2\ \ 3)\}$ is *not* a normal subgroup of S_3. There's no need to check any other $\alpha \in S_3$.

Solutions:

1. **Let K** $= \{\varepsilon, (2 \quad 3)\}$**. Is** $\mathbf{K \triangleleft S_3}$**?**

Method 2: Use the *Normal Subgroup Test*. Show that $\alpha K \alpha^{-1} \subseteq K$ for all $\alpha \in S_3$.

α	ε	$(2 \quad 3)$	$(1 \quad 2)$
$\alpha K \alpha^{-1}$	K	K	$(1 \quad 2)\, K\, (1 \quad 2)$ $= \{\varepsilon, (1 \quad 2)(2 \quad 3)(1 \quad 2)\}$ $= \{\varepsilon, (1 \quad 3)\}$
$\alpha K \alpha^{-1} \subseteq K$?	yes	yes	**NO!**

Since there exists an α such that $\alpha K \alpha^{-1} \not\subseteq K$, K fails the test for normality, and so is not normal.

Solutions:

2. **Prove that $\mathbf{A_n \lhd S_n}$.**

From Thm 5.7, we know that $|A_n| = \dfrac{n!}{2} = \dfrac{|S_n|}{2}$.

From Lagrange's theorem, we know that the number of cosets of A_n in S_n, $[S_n : A_n]$, is $|S_n|/|A_n| = 2$.

We saw Wednesday that whenever $[G : H] = 2$, $H \lhd G$, so $A_n \lhd S_n$.

Solutions:

3. (a) **Let** $\mathbf{H} = <\begin{pmatrix}1 & 2 & 3\end{pmatrix}>$ **in** $\mathbf{S_3}$**, and consider the left cosets of** H**,**

$$\mathbf{S_H} = \{\varepsilon\mathbf{H}, \begin{pmatrix}1 & 2\end{pmatrix}\mathbf{H}\}.$$

Define $\alpha\mathbf{H} * \beta\mathbf{H} \stackrel{\text{def}}{=} \alpha \circ \beta\mathbf{H}$**.**

i. **Find** $\varepsilon\mathbf{H} * \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{H}$**.**

$$\begin{aligned} \varepsilon H * \begin{pmatrix}1 & 3\end{pmatrix} H &\stackrel{def}{=} \left(\varepsilon \circ \begin{pmatrix}1 & 3\end{pmatrix}\right) H = \begin{pmatrix}1 & 3\end{pmatrix} H \\ &= \begin{pmatrix}1 & 2\end{pmatrix} H \end{aligned}$$

Thus

$$\varepsilon H * \begin{pmatrix}1 & 3\end{pmatrix} H = \begin{pmatrix}1 & 2\end{pmatrix} H.$$

We took the coset H and operated it with the coset $\begin{pmatrix}1 & 2\end{pmatrix} H$ – but using $\begin{pmatrix}1 & 3\end{pmatrix}$ as the representative of that coset rather than $\begin{pmatrix}1 & 2\end{pmatrix}$ – and found that the result is the coset $\begin{pmatrix}1 & 2\end{pmatrix} H$.

Solutions:

3.(a)ii. **Find $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$ H $*$ $\begin{pmatrix}1 & 2\end{pmatrix}$ H. Which of the two cosets do you get?**

$$\begin{aligned}\begin{pmatrix}1 & 2 & 3\end{pmatrix} H * \begin{pmatrix}1 & 2\end{pmatrix} H &\overset{def}{=} (\begin{pmatrix}1 & 2 & 3\end{pmatrix} \circ \begin{pmatrix}1 & 2\end{pmatrix})H \\ &= \begin{pmatrix}1 & 3\end{pmatrix} H \\ &= \begin{pmatrix}1 & 2\end{pmatrix} H\end{aligned}$$

Again, we found that the coset H – this time using $\begin{pmatrix}1 & 2 & 3\end{pmatrix}$ as its representative –operated with the coset $\begin{pmatrix}1 & 2\end{pmatrix} H$ is $\begin{pmatrix}1 & 2\end{pmatrix} H$.

Solutions:

3.(a) iii. **Find $(1\ \ 3\ \ 2)\,\mathbf{H} * (2\ \ 3)\,\mathbf{H}$. Which of the two cosets do you get?**

$$\begin{aligned}(1\ \ 3\ \ 2)\,H * (2\ \ 3)\,H &\overset{def}{=} ((1\ \ 3\ \ 2)\circ(2\ \ 3))H \\ &= (1\ \ 3)\,H \\ &= (1\ \ 2)\,H.\end{aligned}$$

Once again, I've got a version of the coset H combined with a version of the coset $(1\ \ 2)\,H$, and once again I've ended up with the coset $(1\ \ 2)\,H$.

Solutions:

3.(a) iv. **Compare your results to the first three questions. What happened? Was it what you expected, or something different?**

In each case, I had $H * (1 \quad 2)\, H$, but written different ways. And each time, I found that the result was the coset $(1 \quad 2)\, H$, although it appeared with different representatives.

Solutions:

3.a. v. **In general, for this subgroup H, show that if $\alpha_1 \mathbf{H} = \alpha_2 \mathbf{H}$ and $\beta_1 \mathbf{H} = \beta_2 \mathbf{H}$, then $\alpha_1 \mathbf{H} * \beta_1 \mathbf{H} = \alpha_2 \mathbf{H} * \beta_2 \mathbf{H}$. (This shows that the operation $*$ is well-defined).**

Assume $\alpha_1 H = \alpha_2 H$ and $\beta_1 H = \beta_2 H$.

$$\begin{aligned}
\alpha_1 H = \alpha_2 H &\implies \exists\ h_1, h_2 \in H \ni \alpha_1 h_1 = \alpha_2 h_2 \\
&\implies \alpha_1 = \alpha_2 h_2 h_1^{-1} \text{ for some } h_1, h_2 \in H.
\end{aligned}$$

$$\begin{aligned}
\beta_1 H = \beta_2 H &\implies \exists\ g_1, g_2 \in H \ni \beta_1 g_1 = \beta_2 g_2 \\
&\implies \beta_1 = \beta_2 g_2 g_1^{-1} \text{ for some } g_1, g_2 \in H
\end{aligned}$$

$$\begin{aligned}
\therefore \alpha_1 H * \beta_1 H &\overset{def}{=} (\alpha_1 \beta_1) H \\
&= [(\alpha_2 h_2 h_1^{-1})(\beta_2 g_2 g_1^{-1})] H \\
&= [\alpha_2 (h_2 h_1^{-1} \beta_2) g_2 g_1^{-1}] H
\end{aligned}$$

Solutions:

Assume $\alpha_1 H = \alpha_2 H$ and $\beta_1 H = \beta_2 H$. $\exists\ h_1, h_2, g_1, g_2 \in H$ such that:

$$\alpha_1 H * \beta_1 H = [\alpha_2 (h_2 h_1^{-1} \beta_2) g_2 g_1^{-1}]H$$

$(h_2 h_1^{-1})\beta_2 \in H\beta_2$.

$H \triangleleft\ S_3 \Rightarrow H\beta_2 = \beta_2 H$, so $h_2 h_1^{-1}\beta_2 = \beta_2 h$ for some $h \in H$.
Thus

$$\begin{aligned} \alpha_1 H * \beta_1 H &= \alpha_2(\beta_2 h) g_2 g_1^{-1} H \\ &= [\alpha_2 \beta_2 (h g_2 g_1^{-1})]H \end{aligned}$$

Since h, g_2, g_1^{-1} are all in H, $hg_2 g_1^{-1} \in H$, so

$$\begin{aligned} \alpha_1 H * \beta_1 H &= \alpha_2 \beta_2 H \\ &\stackrel{def}{=} \alpha_2 H * \beta_2 H \end{aligned}$$

Thus the operation $\alpha H * \beta H = (\alpha\beta)H$ is well-defined.
Benefits: We could now check whether the set of cosets forms a group!

Solutions:

3.(b) **Let** $\mathbf{K} = <\begin{pmatrix}1 & 2\end{pmatrix}>$ **in** $\mathbf{S_3}$**, and consider the set of left cosets of** K

$$\mathbf{S_K} = \{\mathbf{K}, \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{K}, \begin{pmatrix}2 & 3\end{pmatrix}\mathbf{K}\}.$$

Define $\alpha\mathbf{K} * \beta\mathbf{K} \overset{\text{def}}{=} (\alpha \circ \beta)\mathbf{K}$**.**

$$\begin{aligned}
\begin{pmatrix}1 & 3\end{pmatrix} K * \begin{pmatrix}2 & 3\end{pmatrix} K &\overset{def}{=} (\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix})K \\
&= \begin{pmatrix}1 & 3 & 2\end{pmatrix} K \\
\begin{pmatrix}1 & 2 & 3\end{pmatrix} K * \begin{pmatrix}1 & 3 & 2\end{pmatrix} K &\overset{def}{=} (\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 3 & 2\end{pmatrix})K \\
&= \varepsilon K
\end{aligned}$$

Since $\begin{pmatrix}1 & 3 & 2\end{pmatrix} K \neq \varepsilon K$, these two different ways of doing the same operation on the same elements gives different results.
From this, we can conclude that with this choice of subgroup, the operation $*$ is *not* well-defined!