In Class Work

1. What is **the order of** D_4? That is, what is $|D_4|$?

2. In D_4, what is **the order of** the element H (reflection across the horizontal axis)? That is, what is $|H|$? How about the order of the rotation R_{90}, $|R_{90}|$?

3. What is $|GL(2, \mathbb{R})|$?

4. In $\mathbb{Z}_8$, what is $|2|$? How about $|3|$?

Solutions:

1. What is *the order of* D_4? That is, what is $|D_4|$?

   Order of a *group*= the number of elements: $|D_4| = 8$.

2. In D_4, what is *the order of* the element the reflection H, $|H|$? How about the order of the rotation R_{90}, $|R_{90}|$?

   Order of an element = (in this case) smallest # of times we can do the motion to end up equivalent to the identity motion.

   $|H| = 2$, $|R_{90}| = 4$.

3. What is $|GL(2, \mathbb{R})|$?

   Since $GL(2, \mathbb{R})$ is the set of all 2×2 matrices with entries in $\mathbb{R}$ and non-zero determinants, $|GL(2, \mathbb{R})| = \infty$.

Solutions:

4. In $\mathbb{Z}_8$, what is $|2|$? How about $|3|$?

   $2+2 \bmod 8 = 4$, $2+2+2 \bmod 8 = 6$, $2+2+2+2 \bmod 8 = 0$. Thus the smallest number of 2's we can add to get the identity is 4, so $|2| = 4$.

   As for 3, we're not going to get to $3+3+\ldots+3 = 0 \bmod 8$ until we have eight 3's.

   (Check it out!)

   Thus $|3| = 8$.

Cayley Table for D_4

$\circ$	R_0	R_{90}	R_{180}	R_{270}	H	N	V	P
R_0	R_0	R_{90}	R_{180}	R_{270}	H	N	V	P
R_{90}	R_{90}	R_{180}	R_{270}	R_0	P	H	N	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	P	H	N
R_{270}	R_{270}	R_0	R_{90}	R_{180}	N	V	P	H
H	H	N	V	P	R_0	R_{90}	R_{180}	R_{270}
N	N	V	P	H	R_{270}	R_0	R_{90}	R_{180}
V	V	P	H	N	R_{180}	R_{270}	R_0	R_{90}
P	P	H	N	V	R_{90}	R_{180}	R_{270}	R_0

Group Work:

1. Is $\mathbb{Z}$ a subgroup of $\mathbb{R}$, under addition?

2. Is $\mathbb{Z}_5$ a subgroup of $\mathbb{Z}$?

3. Is $U(10)$ a subgroup of $\mathbb{Z}_{10}$?

Solutions:

1. Is $\mathbb{Z}$ a subgroup of $\mathbb{R}$, under addition?

   We already know that both $(\mathbb{R}, +)$ and $(\mathbb{Z}, +)$ are groups.

   Because

   - $\mathbb{Z} \subseteq \mathbb{R}$
   - $\mathbb{Z}$ is a group **under the group operation** of $\mathbb{R}$

   $\mathbb{Z}$ is a subgroup of $\mathbb{R}$ under addition, by the definition of a subgroup.

Solutions:

2. Is $\mathbb{Z}_5$ a subgroup of $\mathbb{Z}$?

   $(\mathbb{Z}_5, + \mod 5)$ is a group and $(\mathbb{Z}, +)$ is a group.

   - $\{0, 1, 2, 3, 4\} \subseteq \mathbb{Z}$
   - The **operations are not the same**.

   Therefore $\mathbb{Z}_5$ is not a subgroup of $\mathbb{Z}$.

Solutions:

2. Is $U(10)$ a subgroup of $\mathbb{Z}_{10}$?

   $U(10)$ is the set of all integers between 1 and 10 (inclusive) relatively prime to 10 $\Longrightarrow U(10) = \{1, 3, 7, 9\}$.

   $\big(U(10), \cdot \mod 10\big)$ is a group and $(\mathbb{Z}_{10}, + \mod 10)$ is a group.

   - $U(10) \subseteq \mathbb{Z}_{10}$
   - The **operations are not the same**.

   Therefore $U(10)$ is not a subgroup of $\mathbb{Z}_{10}$.

Requirements for a Subset to be a Subgroup:

Let G be a group, and suppose that H is a subset of G – that is $H \subseteq G$. In order for H to be a subgroup of G ($H \leq G$), H must be a group under the operation of G.

That is, H must satisfy the following properties:

1. $H \neq \emptyset$.
2. H must be closed under the group operation of G. That is, for all $a, b \in H$, we must have $ab \in H$. (not just in G)
3. H must be associative under the group operation of G
4. H must have an identity element. That is, $e_G \in H$.
5. Every element in H must have an inverse also in H. That is, for all $a \in H$, we must have $a^{-1} \in H$ as well.

The 2-Step Subgroup Test:

Let $(G, *)$ be a group and $H \subseteq G$.
In order to show that $H \leq G$, it suffices to show:

1. *non-empty:* $H \neq \emptyset$
2. *H is closed under G's operation:* For all $a, b \in H$, $ab \in H$.
3. *Every element in H has an inverse again in H:* For all $a \in H$, $a^{-1} \in H$.

Group A		Group B		Group C		Group D	
Dan	Q_4	Alfred	M	Abbe	D_6	Luke	$U(24)^*$
Marie	$\mathbb{C}_9{}^*$	Becky	H	Eric	$\mathcal{T}_\in$	Erin	$\mathbb{Q}_{2,3}$
Aubrie	$\mathbb{C}_{10}$	Stephanie	$GL(2,\mathbb{Z}_2)$	Tiffany	$\mathcal{F}$	Laura	$\mathbb{Z}_2[x]$
Sam	$\mathbb{Z}[i]$	Roy	$\mathcal{F}$	Rebecca	M	Shawn	$U(13)$
Christina	$\mathbb{Z}_2[x]$						