The 2-Step Subgroup Test:

Let $(G, *)$ be a group and $H \subseteq G$.
In order to show that $H \leq G$, it suffices to show:

0. *non-empty:* $H \neq \emptyset$
1. *H is closed under G's operation:* For all $a, b \in H$, $ab \in H$.
2. *Every element in H has an inverse again in H:* For all $a \in H$, $a^{-1} \in H$.

In Class Work

Is

$$\begin{aligned} 3\mathbb{Z} &= \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\} \\ &= \{3k | k \in \mathbb{Z}\} \end{aligned}$$

a subgroup of $\mathbb{Z}$?

Solution:

Is $3\mathbb{Z} = \{3k | k \in \mathbb{Z}\}$ a subgroup of $\mathbb{Z}$?

- $3\mathbb{Z} \neq \emptyset$, $3\mathbb{Z} \subseteq \mathbb{Z}$, and they share the same operation.
  Use the 2-step subgroup test ...
- Closure: The group operation of $\mathbb{Z}$ is addition.
  Let $a, b \in 3\mathbb{Z}$. NTS $a + b \in 3\mathbb{Z}$.

$$\begin{aligned} a, b \in 3\mathbb{Z} &\Longrightarrow \exists n, m \in \mathbb{Z} \ni a = 3n, b = 3m \\ &\Longrightarrow a + b = 3n + 3m = 3(n + m) \end{aligned}$$

  Since $n + m \in \mathbb{Z}$, $a + b \in 3\mathbb{Z}$, so $3\mathbb{Z}$ is closed.
- Inverses: Let $a \in 3\mathbb{Z}$. NTS the inverse of a is in $3\mathbb{Z}$.
  Since $\mathbb{Z}$ is an additive group, we know the inverse of a is $-a$.

$$\begin{aligned} a \in 3\mathbb{Z} &\Longrightarrow a = 3m \text{ for some m } \in \mathbb{Z} \\ &\Longrightarrow -a = -3m = 3(-m) \text{ and } -m \in \mathbb{Z} \\ &\Longrightarrow -a \in 3\mathbb{Z} \end{aligned}$$

Therefore $3\mathbb{Z} \leq \mathbb{Z}$.

Theorem 3.3:

(Finite Subgroup Test) Let G be a *finite* group and let H be a non-empty subset of G. If H is closed under the group operation, then $H \leq G$.

$\boxed{D_4}$

$\circ$	R_0	R_{90}	R_{180}	R_{270}	H	N	V	P
R_0	R_0	R_{90}	R_{180}	R_{270}	H	N	V	P
R_{90}	R_{90}	R_{180}	R_{270}	R_0	P	H	N	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	P	H	N
R_{270}	R_{270}	R_0	R_{90}	R_{180}	N	V	P	H
H	H	N	V	P	R_0	R_{90}	R_{180}	R_{270}
N	N	V	P	H	R_{270}	R_0	R_{90}	R_{180}
V	V	P	H	N	R_{180}	R_{270}	R_0	R_{90}
P	P	H	N	V	R_{90}	R_{180}	R_{270}	R_0

In Class Work

The Cayley table for $U(14) = \{1, 3, 5, 9, 11, 13\}$ is shown below. Find $< 9 >$, $|9|$, and $| < 9 > |$.

$\cdot$ mod 14	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1