$$\boxed{D_4}$$

| $\circ$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $N$ | $V$ | $P$ |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $N$ | $V$ | $P$ |
| $R_{90}$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $P$ | $H$ | $N$ | $V$ |
| $R_{180}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $V$ | $P$ | $H$ | $N$ |
| $R_{270}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $R_{180}$ | $N$ | $V$ | $P$ | $H$ |
| $H$ | $H$ | $N$ | $V$ | $P$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ |
| $N$ | $N$ | $V$ | $P$ | $H$ | $R_{270}$ | $R_0$ | $R_{90}$ | $R_{180}$ |
| $V$ | $V$ | $P$ | $H$ | $N$ | $R_{180}$ | $R_{270}$ | $R_0$ | $R_{90}$ |
| $P$ | $P$ | $H$ | $N$ | $V$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $R_0$ |

# In Class Work

1. The Cayley table for $U(14)$ is shown below.

$\boxed{\text{U(14):}}$

| $\cdot$ mod 14 | 1 | 3 | 5 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 9 | 11 | 13 |
| 3 | 3 | 9 | 1 | 13 | 5 | 11 |
| 5 | 5 | 1 | 11 | 3 | 13 | 9 |
| 9 | 9 | 13 | 3 | 11 | 1 | 5 |
| 11 | 11 | 5 | 13 | 1 | 9 | 3 |
| 13 | 13 | 11 | 9 | 5 | 3 | 1 |

Find the following:

(a) $< 9 >$, $|9|$, $| < 9 > |$
(b) $< 11 >$, $|11|$, $| < 11 > |$
(c) $< 3 >$, $|3|$, $| < 3 > |$

2. In $\mathbb{Z}$, find $< 4 >$.

# Solutions:

1. $\boxed{U(14):}$

| $\cdot$ mod 14 | 1 | 3 | 5 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 5 | 9 | 11 | 13 |
| 3 | 3 | 9 | 1 | 13 | 5 | 11 |
| 5 | 5 | 1 | 11 | 3 | 13 | 9 |
| 9 | 9 | 13 | 3 | 11 | 1 | 5 |
| 11 | 11 | 5 | 13 | 1 | 9 | 3 |
| 13 | 13 | 11 | 9 | 5 | 3 | 1 |

(a)

$$\begin{aligned} <9> &= \{9, 9^2, 9^3, \ldots\} = \{9, 11, 1\} \\ |9| = 3 \quad &; \quad |<9>| = 3 \end{aligned}$$

(b)

$$\begin{aligned} <11> &= \{11, 11^2, 11^3, \ldots\} = \{11, 9, 1\} = <9> \\ |11| = 3 \quad &; \quad |<11>| = 3 \end{aligned}$$

(c)

$$\begin{aligned} <3> &= \{3, 3^2, 3^3, \ldots\} = \{3, 9, 13, 11, 5, 1\} = U(14) \\ |3| = 6 \quad &; \quad |<3>| = 6 \end{aligned}$$

# Solutions:

2. In $\mathbb{Z}$, find $<4>$.

$$<4>= \text{ (since } \mathbb{Z} \text{ additive) } = \{4n | n \in \mathbb{Z}\} = 4\mathbb{Z}.$$

## $\boxed{\text{Subgroups of } \mathbb{Z}_8}$

The following are each subgroups:

$$\begin{aligned}
<0> &= \{0\} \\
<1> &= \{1,2,3,4,5,6,7,0\} = \mathbb{Z}_8 \\
<2> &= \{2,4,6,0\} \\
<3> &= \{3,6,1,4,7,2,5,0\} = \mathbb{Z}_8 \\
<4> &= \{4,0\} \\
<5> &= \{5,2,7,4,1,6,3,0\} = \mathbb{Z}_8 \\
<6> &= \{6,4,2,0\} =< 2 > \\
<7> &= \{7,6,5,4,3,2,1,0\} = \mathbb{Z}_8
\end{aligned}$$

So the only distinct **cyclic** subgroups of $\mathbb{Z}_8$ are:

$$\{0\} \qquad \{0,4\} \qquad \{0,2,4,6\} \qquad \{0,1,2,3,4,5,6,7\}$$

# Question:

Does $\mathbb{Z}_8$ have any non-cyclic subgroups?

## Subgroups of $\mathbb{Z}_8$

If $a, b \in G$, define $< a, b >=$ smallest subgroup containing $a$ and $b$. Does $Z_8$ have any non-cyclic subgroups?

$< 2, 3 >$ : already know that if 3 is in the subgroup, it's all of $\mathbb{Z}_8$

$< 2, 4 >$ : since 2 generates 4, this is just $< 2 >$

$< 2, 5 >$ : already know that if 5 is in the subgroup, it's all of $\mathbb{Z}_8$

$< 2, 6 >$ : since $< 2 >=< 6 >$, this is just $< 2 >$

$< 2, 7 >$ : $\mathbb{Z}_8$

$< 3, 4 >$ : $\mathbb{Z}_8$

$< 3, 5 >$ : $\mathbb{Z}_8$

$< 3, 6 >$ : $\mathbb{Z}_8$

$< 3, 7 >$ : $\mathbb{Z}_8$

$< 4, 5 >$ : $\mathbb{Z}_8$

$< 4, 6 >$ : $< 2 >$

$< 4, 7 >$: $\mathbb{Z}_8$

$< 5, 6 >$: $\mathbb{Z}_8$

$< 5, 7 >$: $\mathbb{Z}_8$

$< 6, 7 >$: $\mathbb{Z}_8$

$D_4$

| $\circ$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $N$ | $V$ | $P$ |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $N$ | $V$ | $P$ |
| $R_{90}$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $P$ | $H$ | $N$ | $V$ |
| $R_{180}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $V$ | $P$ | $H$ | $N$ |
| $R_{270}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $R_{180}$ | $N$ | $V$ | $P$ | $H$ |
| $H$ | $H$ | $N$ | $V$ | $P$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ |
| $N$ | $N$ | $V$ | $P$ | $H$ | $R_{270}$ | $R_0$ | $R_{90}$ | $R_{180}$ |
| $V$ | $V$ | $P$ | $H$ | $N$ | $R_{180}$ | $R_{270}$ | $R_0$ | $R_{90}$ |
| $P$ | $P$ | $H$ | $N$ | $V$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $R_0$ |

# In Class Work

Suppose $G$ is the group defined by the following Cayley table.

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 2 | 1 | 8 | 7 | 6 | 5 | 4 | 3 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 1 | 2 |
| 4 | 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 6 | 5 | 4 | 3 | 2 | 1 | 8 | 7 |
| 7 | 7 | 8 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

1. Find $Z(G)$
2. Find $C(2)$, $C(5)$