Question: If a group G is **not** Abelian, can there exist an element in G that commutes with *all* other elements?

$\boxed{D_4}$

$\circ$	R_0	R_{90}	R_{180}	R_{270}	H	N	V	P
R_0	R_0	R_{90}	R_{180}	R_{270}	H	N	V	P
R_{90}	R_{90}	R_{180}	R_{270}	R_0	P	H	N	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	P	H	N
R_{270}	R_{270}	R_0	R_{90}	R_{180}	N	V	P	H
H	H	N	V	P	R_0	R_{90}	R_{180}	R_{270}
N	N	V	P	H	R_{270}	R_0	R_{90}	R_{180}
V	V	P	H	N	R_{180}	R_{270}	R_0	R_{90}
P	P	H	N	V	R_{90}	R_{180}	R_{270}	R_0

Example

Suppose G is the group defined by the following Cayley table.

$*$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	7	6	5	4	3
3	3	4	5	6	7	8	1	2
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	4	3	2	1	8	7
7	7	8	1	2	3	4	5	6
8	8	7	6	5	4	3	2	1

1. Find $Z(G)$
2. Find $C(2)$, $C(5)$

Examples

Are $U(10) = \{1, 3, 7, 9\}$ and/or $U(12) = \{1, 5, 7, 11\}$ cyclic groups?

$\boxed{U(10):}$

$\cdot$ mod 10	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

$\boxed{U(12):}$

$\cdot$ mod 12	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Comparing Cayley Tables

$U(10) =< 3 >$:

$\cdot \bmod 10$	3^0	3^1	3^2	3^3
3^0	3^0	3^1	3^2	3^3
3^1	3^1	3^2	3^3	3^0
3^2	3^2	3^3	3^0	3^1
3^3	3^3	3^0	3^1	3^2

$\mathbb{Z}_4$:

$+ \bmod 4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$< R_{90} >$:

$\circ$	R_0	R_{90}	R_{180}	R_{270}
R_0	$(R_{90})^0$	$(R_{90})^1$	$(R_{90})^2$	$(R_{90})^3$
R_{90}	$(R_{90})^1$	$(R_{90})^2$	$(R_{90})^3$	$(R_{90})^0$
R_{180}	$(R_{90})^2$	$(R_{90})^3$	$(R_{90})^0$	$(R_{90})^1$
R_{270}	$(R_{90})^3$	$(R_{90})^0$	$(R_{90})^1$	$(R_{90})^2$

$< a >$, where $a^4 = 0$

$*$	e	a	a^2	a^3
e	a^0	a^1	a^2	a^3
a	a^1	a^2	a^3	a^0
a^2	a^2	a^3	a^0	a^1
a^3	a^3	a^0	a^1	a^2

Finding All Groups of Order 4

We already know one group of order 4: the cyclic group

$$< e, a, a^2, a^3 >, \text{ with } |a| = 4.$$

- All cyclic groups of order 4 have this same format.

Finding All Groups of Order 4

Goal: Find all **non-cyclic** groups of order 4.

1. Suppose G is a non-cyclic group of order 4 containing an element a of order 4. What are the four elements of G?

Finding All Groups of Order 4

Goal: Find all **non-cyclic** groups of order 4.

1. Non-cyclic groups of order 4 do not contain any elements of order 4.
2. Let $a \in G$ such that $a \neq e$. What can you say about $|a|$?

Finding All Groups of Order 4

Goal: Find all **non-cyclic** groups of order 4.

1. Non-cyclic groups of order 4 do not contain any elements of order 4.

2. The non-identity elements of G must have order 2 or 3.

3. Suppose $a \in G$ such that $|a| = 3$.
If we can construct such a group, we'll have a new group of order 4. If we can't, we'll know such a group is impossible.
Question: If $a \in G$, what else must be in G?

Finding All Groups of Order 4

3. Let G be a non-cyclic group of order 4, and let $a \in G$, such that $|a| = 3$.

Since G is closed, $a^2, a^3, a^4, \ldots$, *and* $a^{-1}, a^{-2}, \ldots$ must all be in G. Since $|a| = 3$, the only new and distinct element this gives us is a^2.

Thus so far we know that $G = \{e, a, a^2, \boxed{\ ?\ }\}$.

Question: What can you say about the last element? (Think about order!)

Finding All Groups of Order 4

3. Let G be a non-cyclic group of order 4, and let $a \in G$, such that $|a| = 3$.

Since G is closed, $a^2, a^3, a^4, \ldots,$ *and* $a^{-1}, a^{-2}, \ldots$ must all be in G. Since $|a| = 3$, the only new and distinct element this gives us is a^2.

Thus so far we know that $G = \{e, a, a^2, \boxed{\ ?\ }\}$.

The last element, b, can not have order 1 (because then it would be e), 3 (because then not only would b be in G, but so would b^2, and neither b nor b^2 could equal the three already known elements), or 4 or higher (because that introduces too many elements into the group).

Thus the only possibility is for $|b| = 2$.

Question: Can $\{e, a, a^2, b\}$, with $a^3 = e = b^2$ be a group of order 4? (Construct a Cayley table!)

Finding All Groups of Order 4

3. If G is a non-cyclic group of order 4 containing an element a of order 3, then G must have the form $G = \{e, a, a^2, b\}$, with $a^3 = e = b^2$.

$*$	e	a	a^2	b
e	e	a	a^2	b
a	a	a^2	e	$\boxed{?}$
a^2	a^2	e	a	$\boxed{?}$
b	b	$\boxed{?}$	$\boxed{?}$	e

There's nothing left to go in those spots without introducing new elements, but the group must be closed.

Conclusion: There is no non-cyclic group of order 4 that contains any elements of order 3.

Finding All Groups of Order 4

3. Another approach:

If G is a non-cyclic group of order 4 containing an element a of order 3, then G must have the form $G = \{e, a, a^2, b\}$, with $a^3 = e = b^2$.

Since the group is closed, ab must be in G, and so it must be one of the four already listed elements.

That is

$$ab = e \qquad ab = a \qquad ab = a^2 \qquad ab = b.$$

But $ab \neq e$ since $a \cdot a^2 = e$, and inverses are unique
$ab \neq a$ because $b \neq e$
$ab \neq a^2$ because $b \neq a$
$ab \neq b$ because $a \neq e$.

Conclusion: $ab \neq G$, and so there is no non-cyclic group of order 4 that contains any elements of order 3.

Finding All Groups of Order 4

Goal: Find all **non-cyclic** groups of order 4.

1. Non-cyclic groups of order 4 do not contain any elements of order 4.

2. The non-identity elements of G must have order 2 or 3.

3. Order 3 is impossible

4. If any non-cyclic groups of order 4 exist, the three non-identity elements must all have order 2.

Question: We know such a group exists ($U(12)$ and D_2 are examples). But how many fundamentally different ways are there to construct such a table?

Finding All Groups of Order 4

4. Question: How many fundamentally different groups are there with $G = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e$?

$*$	e	a	b	c
e	e	a	b	c
a	a	e	$\boxed{?}$	$\boxed{?}$
b	b	$\boxed{?}$	e	$\boxed{?}$
c	c	$\boxed{?}$	$\boxed{?}$	e

Question: What can you say about ab and ba?

Finding All Groups of Order 4

4. Question: How many fundamentally different groups are there with $G = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e$?

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	$\boxed{?}$
b	b	c	e	$\boxed{?}$
c	c	$\boxed{?}$	$\boxed{?}$	e

ab can't be e, a, or b, so it must be c. Same with ba.

Question: What can you say about ac and ca?

Finding All Groups of Order 4

4. Question: How many fundamentally different groups are there with $G = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e$?

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	$\boxed{?}$
c	c	b	$\boxed{?}$	e

ab can't be e, a, or b, so it must be c. Same with ba.

ac and ca must be b, for the same reasons.

Question: What can you say about bc and cb?

Finding All Groups of Order 4

4. Question: How many fundamentally different groups are there with $G = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e$?

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

ab can't be e, a, or b, so it must be c. Same with ba.

ac and ca must be b, for the same reasons.

bc and cb must be a

Finding All Groups of Order 4

Goal: Find all **non-cyclic** groups of order 4.

Conclusion: There is only type of non-cyclic group of order 4.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

What do you notice about this group?

Finding All Groups of Order 4

Big conclusion: There are only 2 fundamentally different types of groups of order 4: the cyclic group and the group with every non-identity element having order 2. Both of these are Abelian.