Recall:

You found that there are 8 motions on the square that leave the square seemingly unmoved:

$$\{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}.$$

We called these motions the *symmetries of the square*.
Notation:

$$H \circ R_{90} \quad \Longleftrightarrow \quad \text{first rotating counter-clockwise by } 90^\circ$$
$$\text{then reflecting across a horizontal axis.}$$

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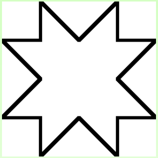
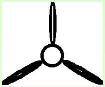
The set of symmetries of an equilateral triangle (3 rotations, 3 reflections) is called D_3, and in general, the set of symmetries of a regular n-gon (n rotations, n reflections) is called D_n.

Consider the following figure:

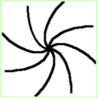


How can we move this and leave it (seemingly) unchanged?

What sort of symmetries does this figure have?



What about this figure?



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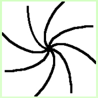

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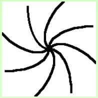

This figure has no reflection symmetry.

It does have 8 rotations: R_{45}, R_{90}, etc. We say this figure has symmetry group $< R_{45} >$.

Below is the *Cayley* table showing the result of applying the operation to any 2 elements.

$\circ$	R_0	R_{90}	R_{180}	R_{270}	H	D	V	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	D	V	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	H	D	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	D'	H	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	V	D'	H
H	H	$\boxed{D^*}$	V	D'	R_0	R_{90}	R_{180}	R_{270}
D	D	V	D'	H	R_{270}	R_0	R_{90}	R_{180}
V	V	D'	H	D	R_{180}	R_{270}	R_0	R_{90}
D'	D'	H	D	V	R_{90}	R_{180}	R_{270}	R_0

* Remember that the D in row H and column R_{90} comes from $H \circ R_{90}$.

Properties of D_4 to focus on:

1. *Closure:* No new motions are introduced. If $A, B \in D_4$, then $A \circ B \in D_4$.
2. *Identity:* R_0 acts as an identity motion — $R_0 \circ A = A \circ R_0 = A$ for all $A \in D_4$.
3. *Inverses:* Every element has an inverse motion that "undoes" what the motion does. For example, $R_{90} \circ R_{270} = R_{270} \circ R_{90} = R_0$.
4. *Associativity:* $(A \circ B) \circ C = A \circ (B \circ C)$ for all $A, B, C \in D_4$.

A few examples of sets, *together with operations*, which are closed, associative, have an identity, and have all necessary inverses are:

- Integers ($\mathbb{Z}$) under $+$
- Rational numbers ($\mathbb{Q}$) under addition
- $\mathbb{Q}$ under multiplication is not a group, as 0 has no inverse –even though every other element does.
- The set of all invertible 2x2 matrices with real entries, under matrix multiplication.

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Examples:

1. $\{x \in \mathbb{Z}^+ | x \leq 200\}$ has a smallest element, by the well-ordering principle.
2. $\{x \in \mathbb{Q}^+ | x \leq 200\}$ doesn't have a smallest element. For any rational number between 0 and 200, you can always find another, smaller, one, by taking half of the one you have. It will of course still be rational.

Division Algorithm:

Let $a, b \in \mathbb{Z}$ where $b > 0$. Then there exist unique integers q, r such that

$$a = qb + r \qquad \text{with} \qquad 0 \leq r < b.$$

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Examples:

If $a = 13$ *and* $b = 5$, *then* $a = 2b + 3$.

Remember, there's nothing in the statement of the division algorithm requiring that you choose a to be the larger of the two integers.

If $a = 5$ *and* $b = 13$, *then* $a = 0 \cdot 13 + 5$.

Theorem: (GCD is a Linear Combination)

For any non-zero $a, b \in \mathbb{Z}$, there exist s and t such that $gcd(a, b) = as + bt$. That is, the $gcd(a, b)$ is a linear combination of a and b. Moreover, $gcd(a, b)$ is the smallest positive integer of the form $as + bt$.

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3. Show $d|a$, $d|b$, so d is a common divisor of a and b.

4. Show $d = gcd(a, b)$, that is, d is the largest of all the common divisors.