*These problems are all either from* Symmetry, Shape and Space*, chapter 6: "Other Dimensions, Other Worlds",* adaptations of problems from that chapter, or inspired by that chapter.

1. (Exercise 9, §6.1) A circle is the set of all points in a plane equidistant from the center. A sphere is the set of all points in space equidistant from the center.

   (a) How would you define a fourth dimensional sphere, called a hypersphere?

   (b) What would we see if a hypersphere passed through our space? Determine this by thinking analogously – what did A. Square observe as the sphere passed through his space?

2. (Expansion of Exercise 14, §6.1) Define a **hyperhypercube** in the 5th dimension analogously to how we defined the hypercube in the 4th:

   > Begin with a hypercube. Place a copy of it parallel to the original in the fifth dimension in "just the right position" (as we do when forming squares and cubes). Attach each vertex of the original hypercube to the corresponding vertex of the copy.

   Using our discussion of the hypercube as inspiration,

(a) How many vertices does the hyperhypercube have, and why? (When answering the why part of the question, discuss this in the context of how the hyperhypercube is formed, rather than simply referring to patterns. Feel free to refer to our work on the hypercube.)

(b) How many edges does the hyperhypercube have, and why? (Again, discuss in the context of how its formed.)

(c) How many 2-dimensional faces does the hyperhypercube have, and why? (Same type of discussion as above.)

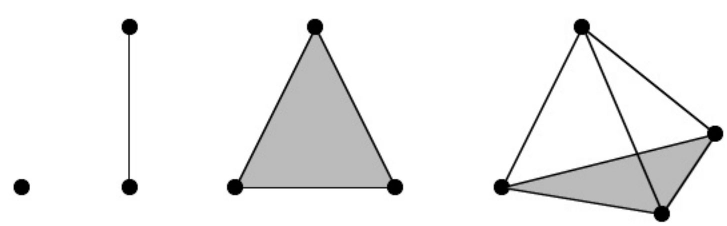
(d) How many 3-dimensional solids does it have? Justify.

(e) How many 4-dimensional regions? Justify.

(f) How many 5-dimensional regions? Justify.

3. (Adaptation of Exercise 14, §6.2) Cubes, hypercubes, hyperhypercubes etc are all higher-dimensional analogies of the two-dimensional square. On page 203-204, your text discusses how other higher-dimensional figures can be built up in analogy with the triangle:

*Start with a single point. In the next generation, add another point above the first, and connect the two, creating a line segment. Place that line segment on the floor and add another point above the line segment and connect this point with each of the points on the line segment, obtaining a triangle. Place the triangle flat on the floor and add another point above it; connecting the new point to each of the points in the triangle gives a three-dimensional solid called the tetrahedron or triangular pyramid.*



> *Adding another point in the fourth dimension,* ana *or* kata *the figure, and connecting this point to each point in the tetrahedron gives a figure called the* hypertetrahedron *or* pentahedroid*.*

Develop analogous formulae to those we developed for the hypercube and the hyperhypercube to fill in the table on the next page.

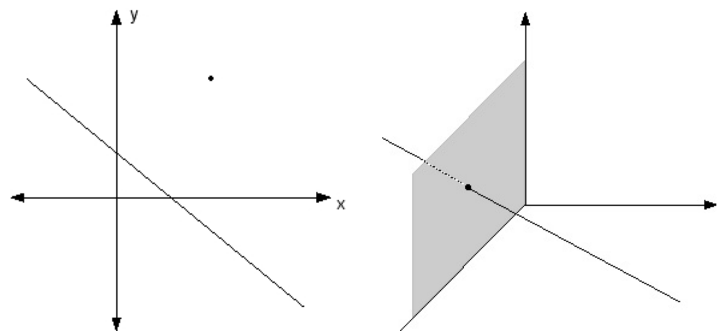
| Dimension | 0D | 1D | 2D | 3D | 4D | 5D |
|---|---|---|---|---|---|---|
| Figure | point | segment | triangle | tetrahedron | hypertetrahedron | hyperhypertetrahedron |
| vertices $v$ | | | | | | |
| edges $e$ | | | | | | |
| faces $f$ | | | | | | |
| solids $s$ | | | | | | |
| 4D regions $t$ | | | | | | |
| 5D regions $u$ | | | | | | |

4. (Exercise 3, §6.2) In the plane, note that any two non-parallel lines intersect in a point.

   (a) Generalize this idea to the intersection of two non-parallel planes in the third dimension. That is, what does the intersection of any two non-parallel planes look like? You should be able to verify this by experiment.

   (b) Generalize even further, this time to the intersection of two non-parallel three-dimensional spaces in hyperspace. For this, of course, you'll need to use analogy to the dimensions you *can* picture.

5. (Exercise 4, §6.2) In the illustration on the left below, you see that in general, (that is, usually) a point and a line on the plane will not intersect at all. In other words, given a randomly chosen line and a randomly chosen point in the same plane, more likely than not the point will not be on the line. In the illustration shown on the right, you see that usually a line and a plane in 3 dimensional space will intersect in a point.




(Of course, this does not always occur: the line could be parallel to the plane, or the line could lie on the plane, but given an randomly chosen line in space and a randomly chosen plane in space, *more often than not* the line will intersect the plane in only one point.)

(a) What is the most likely intersection of two lines in the plane? *Draw a sketch to illustrate.*

(b) What is the most likely intersection of two lines in 3-space? *Draw a sketch to illustrate.*

(c) Look at the results for this exercise, as well as those of part (a) of the previous exercise. In each case, we were looking at the intersection of either a point (0D), a line (1D), or plane (2D) with a line (1D) or plane (2D). Furthermore, in each case, the intersection occurred in either the plane (2D) or space (3D). And in each case, the resulting intersection was either nothing (no dimension), a point (0D), or a line (1D). In Part (b) of the previous exercise, you generalized the intersection of two 3D space in hyperspace.

Use your results to fill in the table on the next page that compare the dimension of the two intersecting objects, the dimension of the space the intersection occurs in, and the dimension of the most likely resulting intersection. (I've filled in the two facts that I gave you.)

| Description of intersection | Dimension of object 1 | Dimension of object 2 | Dimension of space in which intersection occurs | Dimension of intersection |
|---|---|---|---|---|
| Point with line in the plane | 0 | 1 | no intersection | none |
| Line with line in the plane | | | | |
| Line with line in space | | | | |
| Line with plane in space | 1 | 2 | 3 | 0 |
| Plane with plane in space | | | | |
| Space with space in hyperspace (4D) | | | | |

(d) Find a relationship (a formula) that relates the dimension of the resulting intersection with the dimensions of the two intersecting objects and the dimension of the space.

(e) What is the most likely intersection of a line and a plane in hyperspace? Justify your answer using whatever balance of insight into the 4th dimension and the formula you developed that works for you.

(f) What is the most likely intersection of two planes in hyperspace? Again, justify your answer

(g) What is the most likely intersection of a plane and a 3-space in hyperspace?