- Exam 2 will take place on Wednesday March 18 from 6-8pm in Mars SC 1141 (the same room as last time).
  - As with Exam 2, I am *designing* the exam to be a (theoretically) 1 hour exam. However, it will cover more material than the last exam did, so because your brain will have to switch gears more, it may take longer than the first exam.
- If you can not take the exam Wednesday March 18 from 6-8pm, contact me as soon as possible.
- PS 7 is included at the beginning of this study guide, and – as with Exam 1 – will not be due.
- The exam will cover *from* PS 4 *through* PS 7. Specifically, Exam 2 will cover all of Chapter 2 (the Golden Ratio, Gnomons, Fibonacci numbers, etc), as well as Perspective up through the development of the Perspective Theorem.
- The solutions to all of the problem sets are available for you to look at suite outside my office. I will add the solutions to this study guide and to PS 7 to that collection soon.
- I will again be giving you a formula sheet along with your exam. This sheet **will** include:
  - the quadratic formula
  - the Golden Ratio
  - the first several Fibonacci numbers
  - Binet's formula

  Not included, that you should know

  - the formula for finding a Fibonacci number from the previous two
  - the distance formula for the distance between two points in 2-space *and* in 3-space
  - the formulas for calculating the 2D coordinates for the perspective image of a point in 3-space – the Perspective Theorem.
  - the Vanishing Point theorem
- Advice:

- Spread studying over several days. Information sinks in better; if you get frustrated, you can take breaks; if some calamity occurs on the day before the exam, you've already done a fair amount of studying; you can get plenty of sleep the night before the exam; etc
- In an ideal world, the best way to study for a math test is to re-read all the readings (including your notes), summarize the topics we've covered, and re-do as many homework problems as possible.

  If don't have time to do all of that,

  * Your main focus should be to *do* as great a variety of problems as possible.
  * In addiition to doing the problems on this problem set, redo as many problems from problems sets as possible.
  * Reading solutions is not enough.
  * Skim the readings and the notes, emphasizing connections with math and art.
- When you're doing problems, focus on *why* the steps are what they are, and why they make sense. Think about how different problems are connected.
- How long should you study for this? A minimum of 6 hours. If you've struggled with the problem sets, then allow considerably more time.
- If you can not do the problems from start to finish without getting help from friend, tutor, solutions or me, you are not ready. Please note that this does not mean you should *memorize* how to do the problems – the exam will involve similar but not identical ideas. If you *understand* how to do all of these problems as well as all your past homework problems, and can use that understanding to *do* all the problems with no help, then you should be prepared.

• Topics:

- The Extreme and Mean Ratio
  * The geometric concept
  * How the geometric concept leads to the Golden Ratio
  * How we can use the Golden Ratio to find desired lengths for any line cut in Extreme and Mean Ratio
  * Being able to follow statements similar to the statement of the Exteme and Mean Ratio to create new ratios (like the Very Cool Ratio in your problem set).
- Showing two geometric shapes are similar; figuring out how long the side of a geometric figure needs to be for it to be similar to another geometric figure.

- Gnomons
- Rectangles with square gnomons and their connection to the golden ratio. What side of the rectangle such a square gnomon must attach to.
- Golden Rectangles & the Extreme and Mean Ratio in art
- What a Golden Triangle is, what it has to do with φ and what it has to do with gnomons
- Fibonacci numbers
- How the Fibonacci numbers are related to φ
  * sequence of $\dfrac{F_n}{F_{n-1}}$
  * Binet's formula
  * anything else you can think of
- Using Binet's formula
- The distance formulae for points in 2-space and for points in 3-space
- Plotting points in 3-space, and how to leave clues as to where that point lies (the box).
- The relationship between points in 3-space (for instance, as in our cube problems)
- The Perspective Theorem-where it comes from, and how to use it

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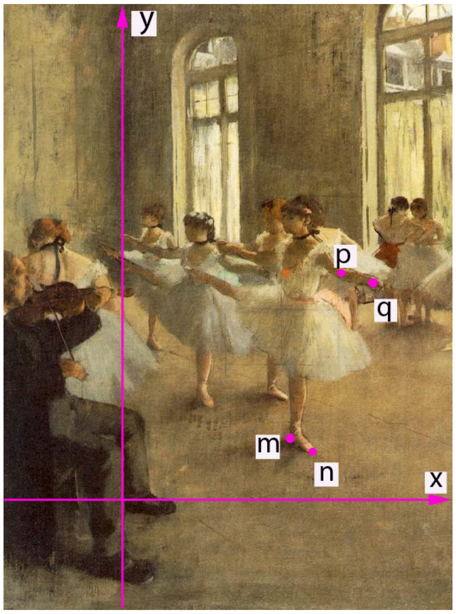
Problem Set 7 Problems:

1. Below is a detail from Edgar Degas' painting, *The Rehearsal*, with 2-dimensional coordinate axes superimposed on the picture plane. Using the points $m(216, 88)$, $n(249, 68)$, $p(283, 302)$, and $q(317, 293)$ in the figure (the coordinates are in pixels), find the following distances:

   (a) $d(m, n)$

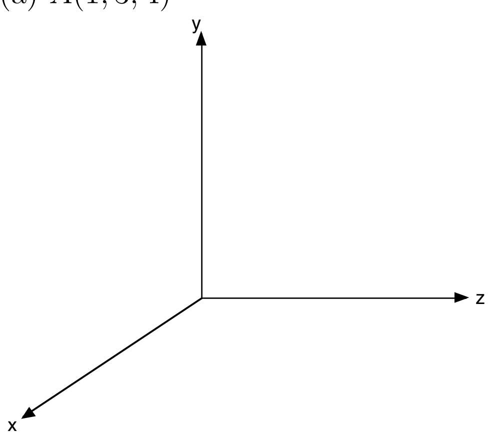
   (b) $d(p, q)$

   (c) Several systems of proportions dictate that a person's foot should be about the same length as their forearm. Thinking of the painting as a window onto a "real" dance studio, it looks as if the actual three-dimensional dancer's left foot and left forearm would be roughly parallel and directly above one another. We'll see later that because of that, if they were indeed the same length in real life, then their *images* in the painting would also be the same length. *Are* these images the same length?

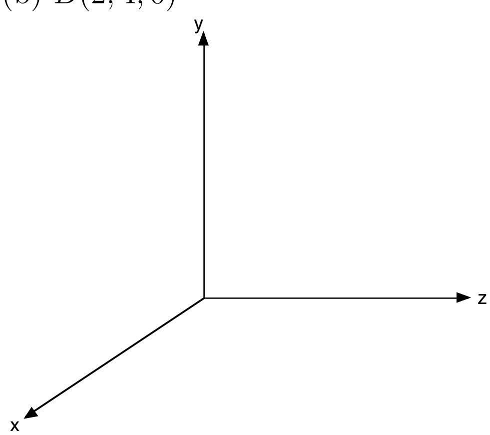


2. Please plot the following points on a set of 3-D coordinate axes (using the coordinate system used for this course rather than the standard coordinate system.) Mark units on your axes, and show enough dashed lines so I can see how you found where to put your points.

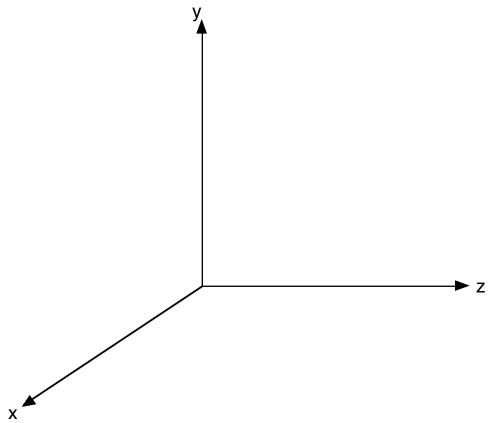
(a) $A(1, 3, 4)$



(b) $B(2, 4, 0)$



(c) $C(0,3,-1)$



3. Assume a viewer (or artist) who is located with one eye on the negative z-axis is looking at two points $A(3,3,2)$ and $B(4,2,7)$.

   (a) Which point is higher?

   (b) Which is closer to the viewer?

   Hint: You may find it helpful to take an omniscient side-view of the situation: draw the axes with the x-axis coming out of the paper as in the previous problems. Sketch in the viewer somewhere along the negative z-axis. Then plot the two points. Since one point is *much* closer to the picture plane than the other, it will be closer to the viewer on the other side of the picture plane, than the other. Which of the two is this closer point?

   (c) Which is further left, to the viewer?

   Hint: For this, you may find it helpful to now draw the axes so that the negative z-axis is coming out of the paper and the positive x-axis is pointing right. Again sketch in the points. Now you are looking at roughly the same thing the viewer is, and so should be able to tell which point is further to the left.

4. In this problem, you're going to be considering a box whose faces are parallel to the coordinate planes. Suppose the corners of this box have the following coordinates:

Bottom	Top
$A(1,3,4)$	$E(1,7,4)$
$B(8,3,4)$	$F(8,7,4)$
$C(8,3,10)$	$G(8,7,10)$
$D(1,3,10)$	$H(1,7,10)$

Note 1: You do not have to plot each of the 8 corners and draw the box in order to do this question, although you certainly can if you think that will help you think about the problem clearly.

Note 2: When using words like "wide" and "deep" below, I am thinking about how the box would appear to a person with their eye at a viewing position on the negative z-axis.

(a) How wide is the box (that is, in the x direction)?

(b) How tall is the box in the y direction?

(c) How deep is the box (that is, how far back in the z direction does it go)?

5. Now I want you to look for patterns in Problem 4, and use them. We again have a box whose faces are parallel to the coordinate planes. Suppose the coordinates of two opposing corners of a box (the front-left-bottom corner and the rear-right-top corner) have coordinates $(3, -1, 5)$ and $(10, 7, 10)$.

(a) How wide is the box, to the viewer?

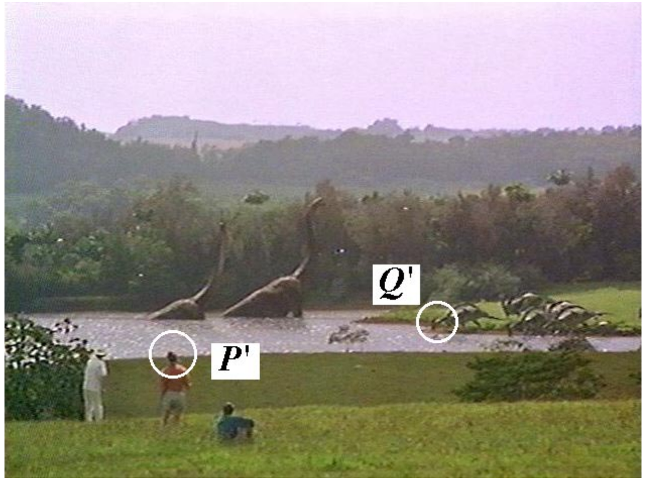
(b) How tall is the box?

(c) How deep is the box, to the viewer? (That is, how far back does it go?)

(d) Use the insights gained from parts (a) through (c) to determine the coordinates of the remaining 6 corners.

6. Think of the Jurassic Park image in Figure 3 of Lesson 2, included below, as being an image of a *real* scene painted onto a picture plane.

- $P(x, y, z)$ = top of woman's head in real 3D life
- $Q(x, y, z)$ = top of the drinking dinosaur's head real life.
- $P'(x, y)$ = *image* of woman's head in the 2D picture
- $Q'(x, y)$ = *image* of drinking dino's head in the 2D picture



Which is bigger:

(a) the x-coordinate of P, or the x-coordinate of Q?

(b) the y-coordinate of P, or the y-coordinate of Q?

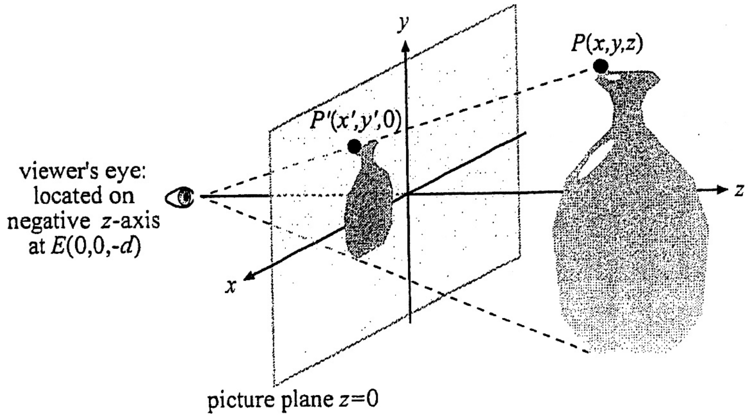
(c) the z-coordinate of P, or the z-coordinate of Q?

(d) the x-coordinate of P', or the x-coordinate of Q'?

(e) the y-coordinate of P', or the y-coordinate of Q'?

(f) the z-coordinate of P', or the z-coordinate of Q'? (This is a trick question)

7. In the figure below (from Lesson 1 of *Lessons in Mathematics and Art*), suppose that $d = 3$ and suppose that the point $P(x, y, z)$ were **moved** so that $x = 0$, $y = 4$, and $z = 5$.



(a) Which coordinate plane would the point $P(x, y, z)$ lie in?

Recall coordinate planes: The xz-plane, determined by the x and z axes, is the horizontal plane through the origin; like the "floor". The xy-plane, determined by the x and y axes, is the vertical plane through the origin; think of it as the picture plane or as a window. The yz-plane, determined by the y and z axes, is the vertical plane through the origin that we might think of a side wall. In the "Artist's View", the xy plane is directly facing us and the yz plane recedes away from us; in the "Omniscient View" (as in this diagram), we see the picture plane off to the side and the yz plane in front of us.

(b) Without using the Perspective Theorem (even once we discuss it in class, I want you to use **ideas** not a theorem), what would x', the image of x in the picture plane, be?

(c) Again without using the Perspective Theorem, what would y', the image of y in the picture plane, be?

Study Guide Problems:

The following problems are intended as a supplement to your review; they are not intended to replace reviewing the reading and class notes, or redoing homework problems.

Remember: You are responsible for all material covered in your reading, whether or not we covered it in class.

1. For each of the following, you'll be drawing a line that is cut in Extreme and Mean Ratio (i.e. the Golden Ratio).

   (a) Suppose we want to draw a line cut in mean and extreme ratio, and we want the longer segment to have length 3. How long should the shorter segment be? Draw such a line as carefully as possible.

   (b) Suppose we want to draw a line cut in mean and extreme ratio, and we want the short segment to be 5 units long. How long would the whole line be? Draw such a line as carefully as possible.

2. Suppose a splinter group in the class neither agrees with Euclid nor with the Very Cool ratio advocates. They feel strongly that the most beautiful way of cutting a line is as follows:

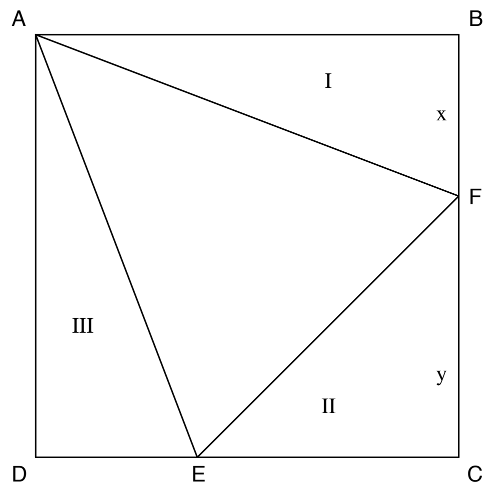
   A line is said to be cut in an **absolutely fabulous** *ratio when the greater segment is to the lesser segment as the whole segment is to twice the greater.*

   (a) What *is* this absolutely fabulous ratio? (That is, find what number it equals)

   (b) Discuss the issue of the short segment versus the long segment.

   Since it turned out our ratio was 1-to-1, the only way that the whole can be to twice the greater as the greater is to the lesser is if the greater and the lesser are equal, and just divide the line in half. In other words, there isn't a "greater" and "lesser" segment, per se.

3. In the following figure, $ABCD$ is a square, and the three triangles I, II, and III have equal areas. By following the steps below, you are going to show that $\frac{y}{x} = \varphi$.



(a) Find the area of triangle I, in terms of x and y.

(b) Using that the areas of triangle I and triangle III are equal, find the length of $\overline{DE}$.

(c) Find the length of $\overline{EC}$.

(d) Find the area of triangle II.

(e) Using that the areas of triangle I and triangle II are equal, show that

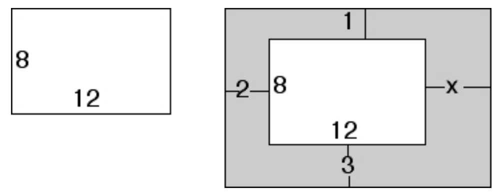
$$\left(\frac{y}{x}\right)^2 - \frac{y}{x} - 1 = 0.$$

(f) Use the quadratic formula to find $\dfrac{y}{x}$.

4. You showed in your homework that a pyramid created by measuring the base using some number of revolutions of the drum, and the height by using twice that number of diameters of the drum is very close to being similar to the Great Pyramid at Gizeh.

   While the Great Pyramid is the most famous of the pyramids, there are others at Gizeh (as well as throughout Egypt). The dimensions for the Second Pyramid at Gizeh are 470.75 feet high, with the sides of the base each being approximately 702' long. Is this pyramid also close to being similar to a pyramid that is constructed using the above technique?

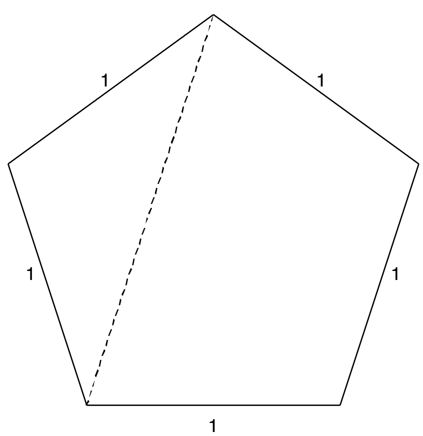
5. Find the value of x so that the shaded "rectangular ring" is a gnomon to the white rectangle.



6. Rectangle A is 2 by 3. Rectangle B is a gnomon to rectangle A. What are the dimensions of rectangle B?

7. A rectangle has a square gnomon. The new rectangle obtained by attaching the square gnomon to the original rectangle has longer leg 20. What are the dimensions of the original rectangle?

8. The regular pentagon in the following figure has sides of length 1. Use the fact that the angle a diagonal forms with the closest side of the pentagon is 36°, along with the results of a problem from PS 6, to show that the length of any one of its diagonals is φ.



9. Use that $F_{26} = 121\,393$ and that $F_{28} = 317\,811$, to find F_{29}.

10. Let a represent the 300th Fibonacci number and b represent the 301st Fibonacci number. Express the 298th Fibonacci number in terms of a and b. Simplify your answer.

11. Claim: $(F_1 + F_2 + F_3 + \ldots + F_N) + 1 = F_{N+2}$. Verify this claim for:

    (a) $N = 4$

(b) $N = 10$

Hint: Verify means show that the claim really *is* true when $N = 4$, or $N = 10$. For instance, to verify that the claim is true for $N = 4$, you need to verify that the left side is equal to the right side. One approach would be to figure out what the left side *is*, and do the same thing for the right side. If they're equal, you have verified the statement.

12. *Calculating powers of φ.*

Remember that φ is one of two solutions to $x^2 - x - 1 = 0$ $\left(\dfrac{1-\sqrt{5}}{2} \text{ is the other}\right)$. Of course, this means that $\varphi^2 - \varphi - 1 = 0$, or in other words, that $\varphi^2 = \varphi + 1$.

(a) Verify that $\varphi^2 = \varphi + 1$ by comparing φ^2 and $\varphi + 1$ on your calculator!

(b) Use that $\varphi^3 = \varphi^2 \cdot \varphi$, along with the above relationship, to show that $\varphi^3 = 2\varphi + 1$. *Hint:* Replace φ^2 with $\varphi + 1$, simplify, and replace φ^2 with $\varphi + 1$ again.

(c) Use your result for φ^3, along with the fact that $\varphi^4 = \varphi^3 \cdot \varphi$, to show that $\varphi^4 = 3\varphi + 2$ – follow a similar strategy as you did in the previous part.

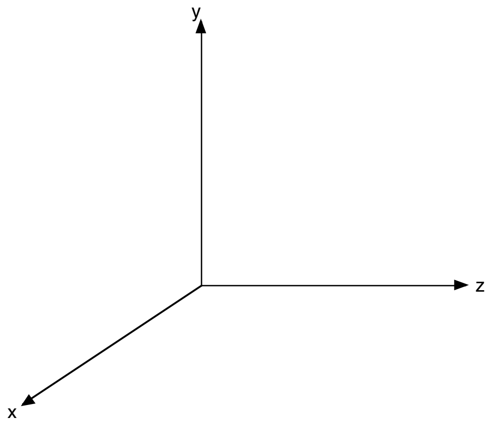
(d) Show that $\varphi^5 = 5\varphi + 3$.

(e) Look for a pattern in the results for φ^2, φ^3, φ^4, and φ^5. Based on what you see, what do you think φ^6 is ? Check your results.

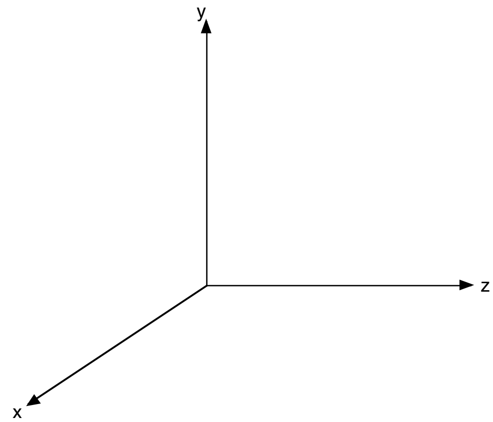
(f) In general, how do you think φ^N can be rewritten, in terms of just a single power of φ and some whole numbers?

13. Plot the following points on a set of 3-D coordinate axes. Leave in lines indicating where in space your points are located.

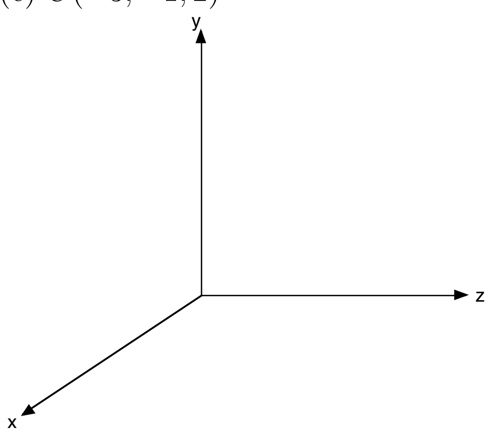
    (a) $A(2, 0, -3)$

    (b) $B(3, 1, 2)$

    (c) $C(-3, -1, 2)$

14. A cube is placed so that its faces are parallel to the coordinate planes. The length of each edge is 7, and to a viewer located on the negative z-axis, the bottom left front corner has coordinates $(1, -3, 2)$.

    (a) What are the coordinates of the other seven corners of the cube?

    (b) Use the Perspective Theorem to find the perspective image of each of the eight corners. Use a viewing distance of 2 units.

    (c) Carefully draw the cube in perspective by using the images of each coordinate that you found in the previous part.