Integration by Substitution:

- Identify u (usually the inside of a composition); use that to find $du = u' \, dx$
- Replace the appropriate expressions of x in the integrand with u and du.
  Note: du can not be in the denominator, raised to any power, or otherwise be inside any function.
- Your substitution has been successful if (1) you now have only u and du – no more x and dx – and (2) if your new integral is something you can integrate.
- Integrate the new simpler function.
- Back substitute: Replace u with its original more complicated formulation in terms of x.

In Class Work

Find the following indefinite integrals, and *check your answers!!*

1. $\displaystyle\int \frac{x}{1+x^2}\,dx \qquad (u = 1 + x^2)$
2. $\displaystyle\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx \qquad (u = \sqrt{x+1})$
3. $\displaystyle\int \sec^2 x\, e^{\tan x}\,dx$
4. $\displaystyle\int 12x\cos(3x^2+5)\,dx$
5. $\displaystyle\int (x^2+4x)(x^3+6x^2+4)^{100}\,dx$
6. $\displaystyle\int \tan(x)\,dx$ HINT: Write $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$
7. $\displaystyle\int \frac{1}{x\ln|x|}\,dx$

Solutions:

1. $\displaystyle\int \frac{x}{1+x^2}\,dx \qquad (u = 1 + x^2)$

- **Notice** The composition is not obvious (you can see it if you write $\frac{x}{1+x^2} = x(1+x^2)^{-1}$). When stuck, look for what portion of the integrand is particularly troublesome.
- Identify u and find du:

$$u = 1 + x^2 \qquad \Rightarrow u' = 2x \qquad \Rightarrow du = 2x\,dx$$

- Replace appropriate expressions of x in the integrand with u and du:

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du.$$

- Integrate the new simpler function:

$$\frac{1}{2}\int \frac{1}{u} = \frac{1}{2}\ln|u| + C$$

- Substitute back in $\displaystyle\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2| + C$

Solutions:

2. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx \qquad (u=\sqrt{x+1})$

- **Notice** Have a double-composition. Chose u to be biggest inside.
- Identify u and find du:

$$u=\sqrt{x+1} \Rightarrow u'=\frac{1}{2}(x+1)^{-1/2} \Rightarrow du=\frac{1}{2}\frac{1}{\sqrt{x+1}}\,dx$$

- Replace appropriate expressions of x in the integrand with u and du:

$$\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx = 2\int e^{\sqrt{x+1}}\frac{1}{2}\frac{1}{\sqrt{x+1}}\,dx = 2\int e^u\,du$$

- Integrate the new simpler function:

$$2\int e^u = 2e^u + C$$

- Back substitute: $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx = 2e^{\sqrt{x+1}} + C.$

Solutions

3. $\int \sec^2 x \, e^{\tan x} \, dx$

- Choose $u =$ inside of composition$= \tan(x)$.
  Then $u' = \sec^2 x$, so $du = \sec^2(x)\, dx$
- Substitute:
$$\int \sec^2 x \, e^{\tan x} \, dx = \int e^u \, du$$
- Integrate:
$$\int \sec^2 x \, e^{\tan x} \, dx = \int e^u \, du = e^u + C$$
- Back-substitute:
$$\int \sec^2 x \, e^{\tan x} \, dx = \int e^u \, du = e^u + C = e^{\tan x} + C$$

Solutions

4. $\int 12x\cos(3x^2+5)\,dx$

- Choose $u =$ inside of composition$= 3x^2+5$.
  Then $u' = 6x$, so $du = 6x\,dx$
- Substitute:

$$\int 12x\cos(3x^2+5)\,dx = 2\int 6x\cos(3x^2+5)\,dx = 2\int \cos(u)\,du$$

- Integrate:

$$\int 12x\cos(3x^2+5)\,dx = 2\int \cos(u)\,du = 2\sin(u)+C$$

- Back-substitute:

$$\int 12x\cos(3x^2+5)\,dx = 2\int \cos(u)\,du = 2\sin(u)+C = 2\sin(3x^2+5)+C$$

Solutions

5. $\int (x^2 + 4x)(x^3 + 6x^2 + 4)^{100}\, dx$

- Choose $u =$ inside of composition$= x^3 + 6x^2 + 4$.
  Then $u' = 3x^2 + 12x$, so $du = (3x^2 + 12x)\, dx = 3(x^2 + 4x)\, dx$
- Substitute:

$$\begin{aligned} \int (x^2 + 4x)(x^3 + 6x^2 + 4)^{100}\, dx &= \frac{1}{3} \int 3(x^2 + 4x)(x^3 + 6x^2 + 4)^{100} \\ &= \frac{1}{3} \int u^{100}\, du \end{aligned}$$

- Integrate:

$$\int (x^2 + 4x)(x^3 + 6x^2 + 4)^{100}\, dx = \frac{1}{3} \int u^{100}\, du = \frac{1}{3(101)} u^{101} + C$$

- Back-substitute:

$$\int (x^2 + 4x)(x^3 + 6x^2 + 4)^{100}\, dx = \frac{1}{303} (x^3 + 6x^2 + 4)^{101} + C$$

Solutions

6. $\int \tan(x)\, dx$ HINT: Write $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$

- Composition isn't obvious, but seen this before, in #1.
- Choose $u = \cos(x)$.
  Then $u' = -\sin(x)$, so $du = -\sin(x)\, dx$
- Substitute:

$$\int \tan(x)\, dx = -\int \frac{-\sin(x)}{\cos(x)}\, dx = -\int \frac{1}{u}\, du$$

- Integrate:

$$\int \tan(x)\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C$$

- Back-substitute:

$$\int \tan(x)\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos(x)| + C$$

Solutions

7. $\int \frac{1}{x \ln|x|}\, dx$

- Again have a fraction; this time, letting u be the entire denominator isn't going to work. Try letting u be just a part (the more complicated part) of the denominator.
- Choose $u = \ln|x|$.
  Then $u' = \frac{1}{x}$, so $du = \frac{1}{x}\, dx$
- Substitute:

$$\int \frac{1}{x \ln|x|}\, dx = \int \frac{1}{x} \frac{1}{\ln|x|}\, dx = \int \frac{1}{u}\, du$$

- Integrate:

$$\int \tan(x)\, dx = \int \frac{1}{u}\, du = \ln|u| + C$$

- Back-substitute:

$$\int \frac{1}{x \ln|x|}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln\Big|\ln|x|\Big| + C$$

In Class Work

1. $\int_0^1 xe^{x^2}\ dx$
2. $\int_0^{(\pi/4)^{1/3}} x^2 \sec(x^3) \tan(x^3)\ dx$
3. $\int_2^4 \frac{1}{x(\ln|x|)^2}\ dx$
4. $\int_1^3 x\sqrt{1+x}\ dx$