Application of Differentiation: Optimization

People need to find when and where maxima and minima occur all the time:

- Maximum revenue
- Minimum cost
- Maximum speed
- Minimum population
- Minimum surface area for a certain volume

In Class Work

Draw graphs with as large a variety of local extrema as you can think of. At each local max or local min, sketch the tangent line. In each case, what can you say about the slope of these tangent lines?

In Class Work

1. Can you draw the graph of a function that has $f'(c) = 0$ at some number c, but that does *not* have a local extremum at $x = c$?
2. Can you draw the graph of a function that has $f'(c)$ undefined at some number $x = c$, but that does *not* have a local extremum at $x = c$

Summary of Today's Conclusions:

- **Definition:** A number c in the domain of a function f is called a **critical number** (or a **critical point**) of f if $f'(c) = 0$ or $f'(c)$ is undefined.
- **Theorem:** Suppose that $f(c)$ is a local extremum (that is, is either a local min or a local max). Then c **must** be a critical number.
- *If* $f(x)$ has a local extremum at $x = c$, then f' **must** be 0 or undefined, but just because f' is zero or undefined doesn't *guarantee* that there's a local extremum at that point.

In Class Work

Find all critical numbers. Use graphing technology to determine whether each is a local minimum, local maximum, or neither.

1. $f(x) = x^4 - 3x^3 + 2$
2. $f(x) = \sin(x)\cos(x)$ on $[0, 2\pi]$
3. $f(x) = x^{1/2}(x^2 - 4)^3$

Solutions:

1. $f(x) = x^4 - 3x^3 + 2$

Need to find: where $f'(x) = 0$ and where $f'(x)$ is undefined.

$$f'(x) = 4x^3 - 9x^2.$$

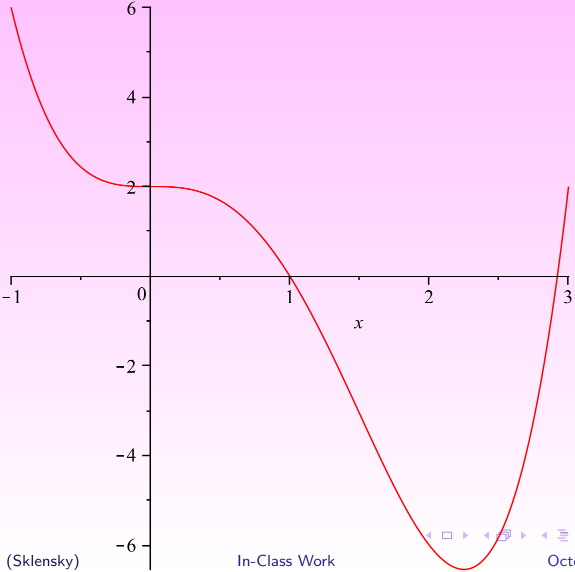
$f'(x)$ is a polynomial and so exists everywhere.

$$\begin{aligned} f'(x) = 0 \quad &\Longrightarrow \quad 4x^3 - 9x^2 = 0 \\ &\Longrightarrow \quad x^2(4x - 9) = 0 \\ &\Longrightarrow \quad x = 0 \text{ or } x = 9/4 \end{aligned}$$

Thus the only critical numbers are $x = 0$ and $x = 9/4$. That is, the only places $f(x)$ can *possibly* have a local max or a local min is at $x = 0$ or at $x = 9/4$.

Solutions

1. (continued)



Solutions

2. $f(x) = \sin(x)\cos(x)$ on $[0, 2\pi]$

Need to find: where $f'(x) = 0$ and where $f'(x)$ is undefined

$$f'(x) = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x).$$

$f'(x)$ is defined everywhere.

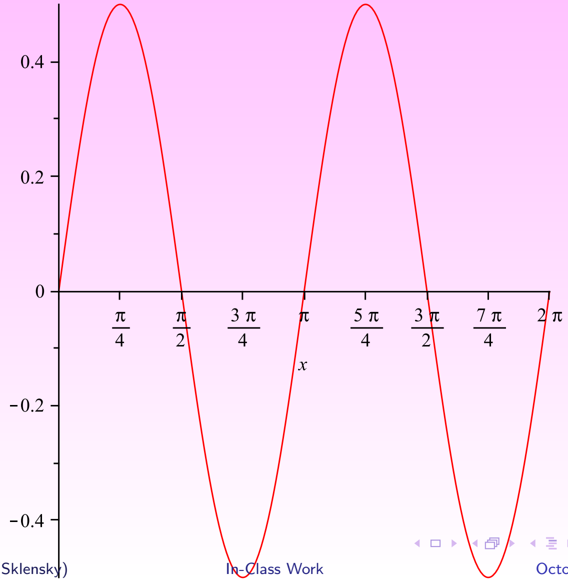
Where does $f'(x) = 0$?

$$\begin{aligned} f'(x) = 0 &\implies \cos^2(x) - \sin^2(x) \\ &\implies \cos^2(x) = \sin^2(x) \\ &\implies \cos(x) = \pm\sin(x) \\ &\implies x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4} \end{aligned}$$

Thus the critical numbers on $[0, 2\pi]$ are $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{7\pi}{4}$, and so the only points on the interval $[0, 2\pi]$ where $f(x) = \sin(x)\cos(x)$ could possibly have a local max or a local min are one of these four points.

Solutions

2. (continued)



3. $f(x) = x^{1/2}(x^2 - 4)^3$

Need to find: where $f'(x) = 0$ and where $f'(x)$ is undefined

$$\begin{aligned} f'(x) &= x^{1/2}\left(3(x^2-4)^2(2x)\right) + \left(\frac{1}{2}x^{-1/2}\right)(x^2-4)^3 \\ &= 6x^{3/2}(x^2-4)^2 + \frac{1}{2x^{1/2}}(x^2-4)^3 \\ &= \frac{1}{2x^{1/2}}(x^2-4)^2\left(12x^2 + (x^2-4)\right) \\ &= \frac{1}{2x^{1/2}}(x^2-4)^2(13x^2-4) \end{aligned}$$

$f'(x)$ is undefined at $x = 0$ (but $f(x)$ *is* defined there).
$f'(x) = 0$ at $x = \pm 2$ and at $x = \pm\frac{2}{\sqrt{13}}$
However: f is not defined for negative values of x.

Thus the critical numbers – the only places f can possibly hope to achieve a local max or local min – are at $x = 0$, $x = 2$, $x = \frac{2}{\sqrt{13}}$

3. (continued)