Summary of Thursday's Conclusions:

- **Definition:** A number c in the domain of a function f is called a **critical number** (or a **critical point**) of f if $f'(c) = 0$ or $f'(c)$ is undefined.

- **Theorem:** Suppose that $f(c)$ is a local extremum (that is, is either a local min or a local max). Then c **must** be a critical number.

- *If* $f(x)$ has a local extremum at $x = c$, then f' **must** be 0 or undefined, but just because f' is zero or undefined doesn't *guarantee* that there's a local extremum at that point.

In Class Work

Find the absolute extrema of the given function on each indicated interval. If you have a graphing calculator available, check your results by looking at a graph.

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$
2. $f(x) = x^{2/3}(x - 5)^3$ on (a) $[-1, 1]$ and (b) $[-1, 8]$

Solutions:

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$

(a) $[-3, 1]$

- **Critical Numbers**: $f'(x) = 4x^3 - 16x$.
  - f' exists everywhere
  -

$$\begin{aligned} f'(x) = 0 &\implies 0 = 4x^3 - 16x \\ &\implies x^3 - 4x = 0 \\ &\implies x(x^2 - 4) = 0 \\ &\implies x = 0, x = 2, x = -2 \end{aligned}$$

The critical numbers are $x = -2$, $x = 0$, and $x = 2$.

Solutions:

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$
(a) $[-3, 1]$

- **Critical Numbers:** $x = 0, x = 2, x = -2$. Only -2 and 0 lie in this interval.
- **Compute** f:
  - $f(-3) = (-3)^4 - 8(-3)^2 + 2 = 81 - 72 + 2 = 11$
  - $f(-2) = (-2)^4 - 8(-2)^2 + 2 = 16 - 32 + 2 = -14$
  - $f(0) = 0^4 - 8(0)^2 + 2 = 2$
  - $f(1) = (1)^4 - 8(1)^2 + 2 = 1 - 8 + 2 = -5$
- **Compare:**
  On the interval $[-3, 1]$, f attains an absolute maximum value of $y = 11$ at $x = -3$ and an absolute mininum value of $y = -14$ at $x = -2$.

Solutions:

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$

(b) $[-1, 3]$

- **Critical Numbers:** Same as in (a): $x = -2, 0, 2$. Only 0 and 2 lie in this interval.
- **Compute** f
  - $f(-1) = (-1)^4 - 8(-1)^2 + 2 = 1 - 8 + 2 = -5$
  - $f(0) = 2$
  - $f(2) = 2^4 - 8(2)^2 + 2 = -14$
  - $f(3) = 3^4 - 8(3)^2 + 2 = 11$
- **Compare:**
  On $[-1, 3]$, f attains an absolute maximum value of $y = 11$ at $x = 3$ and an absolute mininum value of $y = -14$ at $x = 2$.

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1,1]$ and (b) $[-1,8]$

(a) $[-1,1]$

- **Critical Numbers**

$$\begin{aligned} f'(x) &= x^{2/3}\left[3(x-5)^2\right] + \frac{2}{3}x^{-1/3}(x-5)^3 \\ &= \frac{1}{x^{1/3}}\left(x^1\left[3(x-5)^2\right] + \frac{2}{3}(x-5)^3\right) \\ &= \frac{(x-5)^2}{x^{1/3}}\left(3x + \frac{2}{3}(x-5)\right) \\ &= \frac{(x-5)^2}{x^{1/3}}\left(3x + \frac{2}{3}x - \frac{10}{3}\right) \\ &= \frac{(x-5)^2\left(\frac{11}{3}x - \frac{10}{3}\right)}{x^{1/3}} \end{aligned}$$

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1,1]$ and (b) $[-1,8]$

(a) $[-1,1]$

- **Critical Numbers**
  - $f'(x) = \dfrac{(x-5)^2\left(\frac{11}{3}x - \frac{10}{3}\right)}{x^{1/3}}$
  - f' does not exist at $x = 0$, but f does
  - $f' = 0$ when $x = 5$ or when $\dfrac{11}{3}x = \dfrac{10}{3}$, that is when $x = \dfrac{10}{11}$.
  - Thus the critical numbers are $x = 0, \frac{10}{11}, 5$. Only $x = 0$ and $x = 10/11$ lie in this interval.

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1,1]$ and (b) $[-1,8]$
(a) $[-1,1]$

- **Critical Numbers**: $x = 0, \frac{10}{11}, 5$. Only $x = 0$ and $x = 10/11$ lie in $[-1,1]$.
- **Compute**
  - $f(-1) = ((-1)^2)^{1/3}(-1-5)^3 = (-6)^3 = -216$
  - $f(0) = (0)^{2/3}(0-5)^3 = 0$
  - Using a calculator, $f(10/11) \approx -64.25$
  - $f(1) = 1^{2/3}(1-5)^3 = (-4)^3 = -64$
- **Compare**
  On $[-1,1]$, f has absolute maximum value of 0 at $x = 0$ and an an absolute mininum value of $y = -216$ at $x = -1$.

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1, 1]$ and (b) $[-1, 8]$

(b) $[-1, 8]$

- **Critical Numbers**: Still $x = 0, \frac{10}{11}, 5$; all lie in our interval.
- **Compute**
  - $f(-1) = -216$
  - $f(0) = 0$
  - $f(10/11) \approx -64.25$
  - $f(5) = (5)^{2/3}(5-5)^3 = 0$
  - $f(8) = (8)^{2/3}(8-5)^3 = (2)^2(3)^3 = (4)(27) = 108$
- **Compare**

  On $[-1, 8]$, f has absolute maximum value of 108 at $x = 8$ and an an absolute mininum value of -216 at $x = -1$.