In Class Work

1. If $\lim_{x \to a} f(x) = 2$, $\lim_{x \to a} g(x) = -3$, & $\lim_{x \to a} h(x) = 0$, determine the limits:

   (a) $\lim_{x \to a} \left[2f(x) - 3g(x)\right]$

   (b) $\lim_{x \to a} \left[3f(x)g(x)\right]$

   (c) $\lim_{x \to a} \left\{\dfrac{f(x) + g(x)}{h(x)}\right\}$

   (d) $\lim_{x \to a} \left\{\dfrac{3f(x) + 2g(x)}{h(x)}\right\}$

Solutions:

1. If $\lim_{x \to a} f(x) = 2$, $\lim_{x \to a} g(x) = -3$, & $\lim_{x \to a} h(x) = 0$, determine the limits:

   (a) $\lim_{x \to a} \left[2f(x) - 3g(x)\right]$

   $$\begin{aligned} \lim_{x \to a} \left[2f(x) - 3g(x)\right] &= 2 \lim_{x \to a} f(x) - 3 \lim_{x \to a} g(x) \\ &= 2(2) - 3(-3) = 4 + 9 = 13. \end{aligned}$$

   (b) $\lim_{x \to a} \left[3f(x)g(x)\right]$

   $$\begin{aligned} \lim_{x \to a} \left[3f(x)g(x)\right] &= 3 \lim_{x \to a} f(x) \lim_{x \to a} g(x) \\ &= 3(2)(-3) = -18. \end{aligned}$$

1. (continued) If $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -3$, & $\lim_{x\to a} h(x) = 0$, determine the limits:

   (c) $\lim_{x\to a}\left\{\dfrac{f(x)+g(x)}{h(x)}\right\}$

   - Numerator: $\lim_{x\to a}\left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = 2-3 = -1$
   - Denominator: $\lim_{x\to a} h(x) = 0$

   What happens when the numerator is approaching -1 and the denominator approaches 0?

   To get a feel for this, consider what happens when the numerator **is fixed** at -1 while the denominator approaches 0 from the positive side:

$$\frac{-1}{2} = -0.5 \qquad \frac{-1}{1} = -1 \qquad \frac{-1}{0.1} = -10$$

$$\frac{-1}{0.01} = -100 \qquad \frac{-1}{0.001} = -1000 \qquad \frac{-1}{0.0001} = -10000$$

   In general, if the numerator approaches a finite non-zero number while the denominator approaches 0, the limit does not exist.

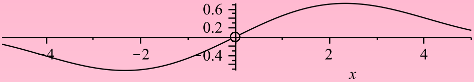
1. (continued) If $\lim_{x \to a} f(x) = 2$, $\lim_{x \to a} g(x) = -3$, & $\lim_{x \to a} h(x) = 0$, determine the limits:

   (d) $\lim_{x \to a} \left\{ \frac{3f(x) + 2g(x)}{h(x)} \right\}$

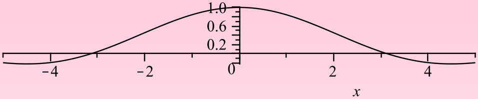
   - Numerator: $\lim_{x \to a} (3f(x) + 2g(x)) = 3 \lim_{x \to a} f(x) + 2 \lim_{x \to a} g(x) = 3(2) + (2)(-3) = 0$
   - Denominator $\lim_{x \to a} h(x) = 0$

In this case, we don't know what happens – this limit is in **indeterminate form**, and evaluating it (at this point) is not possible.

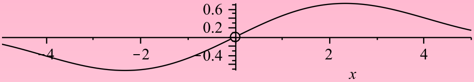
Graph of $y = \dfrac{1 - \cos(x)}{x}$



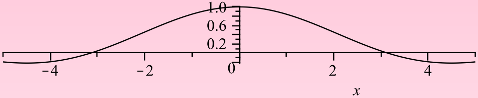
Graph of $y = \dfrac{\sin(x)}{x}$



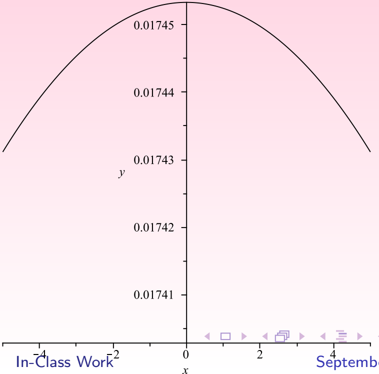
Graph of $y = \dfrac{1 - \cos(x)}{x}$



Graph of $y = \dfrac{\sin(x)}{x}$



Graph of $y = \dfrac{\sin(x)}{x}$
measured in degrees

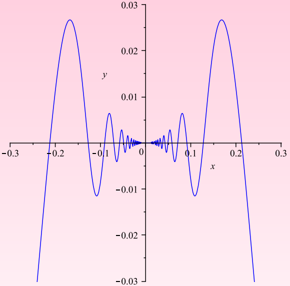


Example 1.3.8: Using the Squeeze Theorem to Verify the Value of a Limit

Determine the value of $\lim_{x \to 0}\left[x^2 \cos\left(\frac{1}{x}\right)\right]$.

Solution:

$\vdots$



x	$x^2\cos(1/x)$
± 0.1	-0.008
± 0.01	8.6×10^{-5}
± 0.001	5.6×10^{-7}
± 0.0001	-9.5×10^{-9}
± 0.00001	-9.99×10^{-100}

The graph and the table of function values suggest the conjecture

$$\lim_{x \to 0}\left[x^2 \cos\left(\frac{1}{x}\right)\right] = 0,$$

which we prove using the Squeeze Theorem.