Integration by Substitution:

- If integrand= product and one factor =composition, then ask :
  Could the integrand come from differentiating using the Chain Rule?
  (Sometimes, it won't be obvious that you have a composition.)
- Choose u to be the inside function in the composition (if there is one).
- Use your choice of u to find du
- Substitute in u and du. Adjust by mult. constant if necessary.

  Remember: du can not be in the denominator, raised to any power, or otherwise be inside any function.
- If this is not possible, or if after doing this there are still some remaining x terms, then your choice of u will not work; try another choice of u, or another method of integration.
- Antidifferentiate the resulting function of u.
- Substitute back in for x

In Class Work

Use substitution to find the following indefinite integrals, and *check your results by differentiating*.

1. $\int \cos(x)e^{\sin(x)}\,dx \qquad (u = \sin(x))$
2. $\int x\sin(\pi x^2)\,dx \qquad (u = \pi x^2)$
3. $\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1-x)$
4. $\int_2^5 \frac{1}{x\ln(x)}\,dx \qquad (u = \ln(x))$
5. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx \qquad (u = \sqrt{x+1})$
6. $\int_0^{\pi/4} \tan(x)\,dx \qquad (u = \cos(x))$
7. $\int_{-\pi/4}^{\pi/3} \cos(3x)\,dx$
8. $\int x^2\sec(x^3)\tan(x^3)\,dx$
9. $\int \frac{\sec^2(x)}{\tan(x)}\,dx$
10. $\int_0^{\pi/4} \sin(x)\cos(x)\,dx$
11. $\int e^x\cos(e^x)\,dx$
12. $\int_1^5 x\sqrt{x-1}\,dx$

Solutions:

1. $\int \cos(x)e^{\sin(x)}\,dx \qquad (u = \sin(x))$

   - Integrand= product; one factor =composition. Came from chain rule?
   - Choose u to be the inside function in the composition (if there is one). Composition: $e^{\sin(x)}$, Inside function: $u = \sin(x)$.
   - Find du:
     Since $u = \sin(x)$, then $\frac{du}{dx} = \cos(x)$, so $du = \cos(x)\,dx$.
   - Substitute in u and du, without changing integral

     $$\int \cos(x)e^{\sin(x)}\,dx = \int e^u\,du$$

   - Antidifferentiate

     $$\int \cos(x)e^{\sin(x)}\,dx = \int e^u\,du = e^u + C$$

   - Substitute back in for x

     $$\int \cos(x)e^{\sin(x)}\,dx = e^{\sin(x)} + C$$

Solutions:

2. $\int x \sin(\pi x^2)\, dx \qquad (u = \pi x^2)$

- Integrand= product; one factor =composition. Came from chain rule?
- Choose u to be the inside function in the composition (if there is one).
  Composition: $\sin(\pi x^2)$, Inside function: $u = \pi x^2$.
- Find du:
  Since $u = \pi x^2$, then $\frac{du}{dx} = 2\pi x$, so $du = 2\pi\, dx$.
- Substitute in u and du, without changing integral

$$\int x \sin(\pi x^2)\, dx = \frac{1}{2\pi} \int \sin(\pi x^2)(2\pi x\, dx) = \frac{1}{2\pi} \int \sin(u)\, du$$

- Antidifferentiate

$$\int x \sin(\pi x^2)\, dx = \frac{1}{2\pi} \int \sin(u)\, du = \frac{1}{2\pi}\big(-\cos(u)\big) + C$$

- Substitute back in for x

$$\int x \sin(\pi x^2)\, dx = -\frac{1}{2\pi} \cos(\pi x^2) + C$$

Solutions:

3. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1 - x)$

- Integrand= product; one factor =composition. Came from chain rule?
- Choose u to be the inside function in the composition (if there is one).
  Composition: $\sqrt{1-x} = (1-x)^{1/2}$, Inside function: $u = 1 - x$.
- Find du:
  Since $u = 1 - x$, then $\dfrac{du}{dx} = -1$, so $du = -1\,dx$.
- Substitute in u and du, without changing integral

$$\int \frac{1}{\sqrt{1-x}}\,dx = -1\int (1-x)^{-1/2}(-1\,dx) = -1\int u^{-1/2}\,du$$

- Antidifferentiate

$$\int \frac{1}{\sqrt{1-x}}\,dx = -1\int u^{-1/2}\,du = (-1)(2)u^{1/2} + C$$

- Substitute back in for x

$$\int \frac{1}{\sqrt{1-x}}\,dx = -2\sqrt{1-x} + C$$

In Class Work

Use substitution to find the following indefinite integrals, and *check your results by differentiating*.

1. $\int \cos(x) e^{\sin(x)}\, dx \qquad (u = \sin(x))$
2. $\int x \sin(\pi x^2)\, dx \qquad (u = \pi x^2)$
3. $\int \frac{1}{\sqrt{1-x}}\, dx \qquad (u = 1 - x)$
4. $\int_2^5 \frac{1}{x \ln(x)}\, dx \qquad (u = \ln(x))$
5. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\, dx \qquad (u = \sqrt{x+1})$
6. $\int_0^{\pi/4} \tan(x)\, dx \qquad (u = \cos(x))$
7. $\int_{-\pi/4}^{\pi/3} \cos(3x)\, dx$
8. $\int x^2 \sec(x^3) \tan(x^3)\, dx$
9. $\int \frac{\sec^2(x)}{\tan(x)}\, dx$
10. $\int_0^{\pi/4} \sin(x) \cos(x)\, dx$
11. $\int e^x \cos(e^x)\, dx$
12. $\int_1^5 x\sqrt{x-1}\, dx$

Solutions:

4. $\displaystyle\int_2^5 \frac{1}{x\ln(x)}\,dx$

Let $u = \ln(x)$. Then $\dfrac{du}{dx} = \dfrac{1}{x}$, so $du = \dfrac{1}{x}\,dx$.

$$\int_2^5 \frac{1}{x\ln(x)}\,dx = \int_2^5 \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx = \int_{x=2}^{x=5} \frac{1}{u}\,du.$$

No x's or dx's left, so a successful substitution!

When $x = 2$, $u = \ln(2)$, and when $x = 5$, $u = \ln(5)$, so

$$\int_2^5 \frac{1}{x\ln(x)}\,dx = \int_{\ln(2)}^{\ln(5)} \frac{1}{u}\,du.$$

$$\int_{\ln(2)}^{\ln(5)} \frac{1}{u}\,du = \ln|u|\Big|_{\ln(2)}^{\ln(5)} = \ln\big(\ln(5)\big) - \ln\big(\ln(2)\big)$$

Solutions:

5. $\displaystyle\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx$

Let $u = \sqrt{x+1}$. Then $\dfrac{du}{dx} = \dfrac{1}{2}(x+1)^{-1/2} \cdot 1 = \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x+1}}$

$du = \dfrac{1}{2}\dfrac{1}{\sqrt{x+1}}\,dx.$

Notice that:

$$\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx = \int e^{\sqrt{x+1}} \cdot 2 \cdot \frac{1}{2}\frac{1}{\sqrt{x+1}}\,dx.$$

If we don't replace the $\sqrt{x+1}$ in the denominator with a u, but instead use it as part of du, this substitution will work.

$$\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx = \int e^{u} \cdot 2\,du.$$

No x's left - a successful substitution.

Solutions:

5. $\displaystyle\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx$

   With $u = \sqrt{x+1}$ and $du = \dfrac{1}{2}\dfrac{1}{\sqrt{x+1}}\,dx$,

$$\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx = \int e^u \cdot 2\,du.$$

$$\begin{aligned}\int e^u \cdot 2\,du &= 2\int e^u\,du \\ &= 2e^u + C \\ &= 2e^{\sqrt{x+1}} + C\end{aligned}$$

Solutions:

6. $\displaystyle\int_0^{\pi/4} \tan(x)\,dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx$

   Let $u = \cos(x)$. Then $\dfrac{du}{dx} = -\sin(x)$, so $du = -\sin(x)\,dx$.

   When $x = 0, u = \cos(0) = 1$. When $x = \pi/4$, $u = \cos(\pi/4) = \frac{\sqrt{2}}{2}$.

$$\begin{aligned}
\int_0^{\pi/4} \tan(x)\,dx &= \int_0^{\pi/4} \frac{1}{\cos(x)}(-1)(-1)\sin(x)\,dx \\
&= \int_1^{\sqrt{2}/2} \frac{1}{u}\,du \\
&= -\ln|u|\Big|_1^{\sqrt{2}/2} \\
&= -\ln\left(\sqrt{2}/2\right) + 0
\end{aligned}$$

Solutions:

7. $\displaystyle\int_{-\pi/4}^{\pi/3} \cos(3x)\, dx$

Let $u = 3x$. Then $\dfrac{du}{dx} = 3$, so $du = 3\, dx$

When $x = -\pi/4$, $u = -3\pi/4$. When $x = \pi/3$, $u = \pi$.

$$\begin{aligned}
\int_{-\pi/4}^{\pi/3} \cos(3x)\, dx &= \int_{-\pi/4}^{\pi/3} \cos(3x) \cdot \frac{1}{3} \cdot 3\, dx \\
&= \frac{1}{3} \int_{-3\pi/4}^{\pi} \cos(u)\, du \\
&= \frac{1}{3} \sin(u) \Big|_{-3\pi/4}^{\pi} \\
&= \frac{1}{3} \Big[\sin\big(\pi\big) - \sin\big(- 3\pi/4\big) \Big]
\end{aligned}$$

Solutions:

8. $\int x^2 \sec(x^3) \tan(x^3)\ dx$

Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$, so $du = 3x^2\ dx$.

$$\begin{aligned}
\int x^2 \sec(x^3)\tan(x^3)\ dx &= \int \sec(x^3)\tan(x^3) \cdot \frac{1}{3} \cdot 3x^2\ dx \\
&= \int \sec(u)\tan(u) \cdot \frac{1}{3}\ du \\
&= \frac{1}{3} \int \sec(u)\tan(u)\ du \\
&= \frac{1}{3}\sec(u) + C \\
&= \frac{1}{3}\sec(x^3) + C
\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.

Solutions:

9. $\displaystyle\int \frac{\sec^2(x)}{\tan(x)}\,dx$

   Tricky: The obvious composition is $\sec^2(x) = (\sec(x))^2$.

   And yet division is also a composition: we have $\big(\tan(x)\big)^{-1}$

   So there's definitely composition going on, but what u should be isn't immediately obvious ... at least not when thinking about what the inside function is.

   Instead, look to see if any piece looks like the derivative of any *other* piece. When we look at it that way, we see immediately that we have a $\tan(x)$, and we have $\sec^2(x) = \dfrac{d}{dx}(\tan(x))$.

   That suggests that we might try letting $u = \tan(x)$. Of course, we don't know if it's going to work yet, but it's a place to start.

Solutions:

9. $\displaystyle\int \frac{\sec^2(x)}{\tan(x)}\,dx$

Let $u = \tan(x)$. Then $\dfrac{du}{dx} = \sec^2(x)$, so $du = \sec^2(x)\,dx$.

Substituting in, we get

$$\begin{aligned}\int \frac{(\sec(x))^2}{\tan(x)}\,dx &= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln\big|\tan(x)\big| + C\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.

Solutions:

10. $\int_0^{\pi/4} \sin(x)\cos(x)\, dx$

Again, tricky: a product, but neither term is obviously a composition.

Not a sum of *two* products, so unlikely to have come from prod rule.

Choose u so du also appears.

$\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$, which is also in integrand.

Let $u = \sin(x)$. Then $\frac{du}{dx} = \cos(x)$, so $du = \cos(x)\, dx$.

$$\begin{aligned} \int_0^{\pi/4} \sin(x)\cos(x)\, dx &= \int_0^{\sqrt{2}/2} u\, du \\ &= \frac{1}{2}u^2 \Big|_0^{\sqrt{2}/2} \end{aligned}$$

Solutions:

11. $\int e^x \cos(e^x)\ dx$

Let $u = e^x$. Then $\frac{du}{dx} = e^x$, so $du = e^x\ dx$.

$$\begin{aligned} \int e^x \cos(e^x)\ dx &= \int \cos(e^x) \cdot e^x\ dx \\ &= \int \cos(u)\ du \\ &= \sin(u) + C \\ &= \sin(e^x) + C \end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.