## Example:

Suppose we need to find $\int_1^2 e^{-x^2}\,dx$.

We can't just find an antiderivative of $e^{-x^2}$ and then plug in the limits. So then what?

Approximate it!

# Example:

Suppose we need to find $\displaystyle\int_1^2 e^{-x^2}\,dx$.

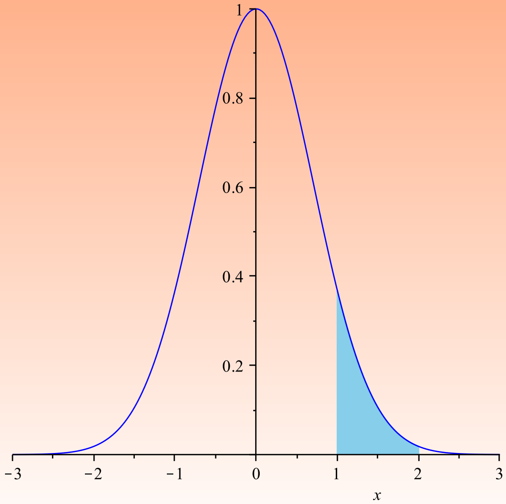
We can't just find an antiderivative of $e^{-x^2}$ and then plug in the limits. So then what?

Approximate it!

- One way: Find a Taylor polynomial for $e^{-x^2}$, and then antidifferentiate that.

$$
\begin{aligned}
e^{-x^2} &\approx 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!} \\
\int_1^2 e^{-x^2}\,dx &\approx \int_1^2 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!}\,dx \\
&\approx \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^6}{6\cdot 3!} + \frac{x^9}{9\cdot 4!} - \frac{x^{11}}{11\cdot 5!} + \frac{x^{13}}{13\cdot 6!}\right]_1^2
\end{aligned}
$$

# Example: Approximating $\int_1^2 e^{-x^2}\,dx$.



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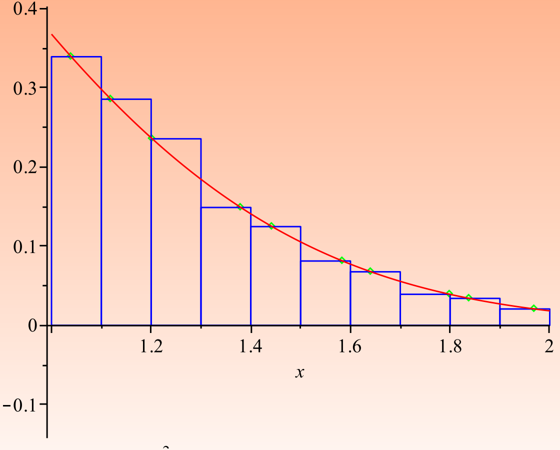

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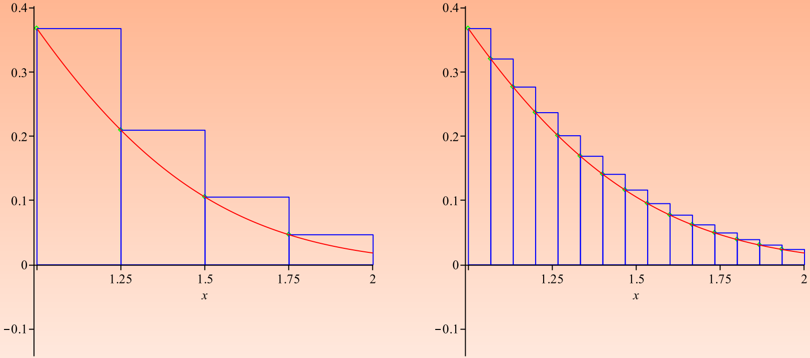
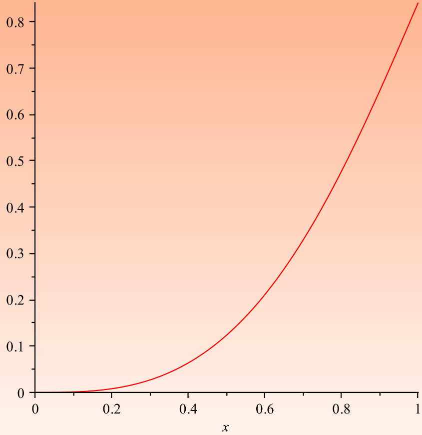


# In Class Work

Let $I = \int_0^1 x \sin(x^2)\ dx$

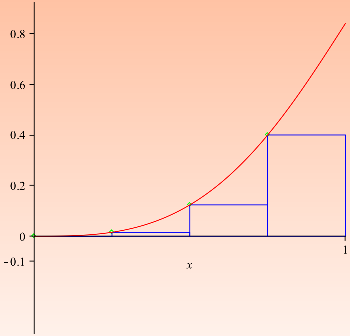



1. Draw a sketch of $L_4$, and then calculate $L_4$.
   Based on your sketch, does $L_4$ overestimate or underestimate $I$?
2. Repeat the above for $R_4$.
3. Sketch $L_{10}$ and $R_{10}$.
   How does $I$ compare to $L_{10}$ and $R_{10}$? That is, are they overestimates? underestimates?
4. Find the exact value of $I$ by using $u$-substitution. Does your result agree with your previous answers?

# Solutions

Let $I = \int_0^1 x \sin(x^2)\ dx$

1. Draw a sketch of $L_4$, and then calculate $L_4$.
   Based on your sketch, does $L_4$ overestimate or underestimate $I$?




$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

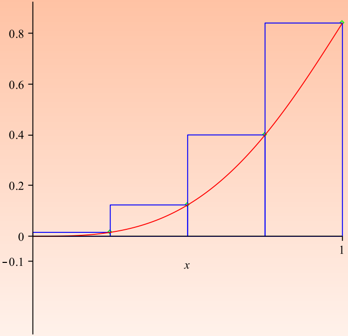
$$\begin{aligned} L_4 &= \frac{1}{4}f(0) + \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{2}{4}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) \\ &= \frac{1}{4}\left(0 + \frac{1}{4}\sin\left(\frac{1}{16}\right) + \frac{2}{4}\sin\left(\frac{4}{16}\right) \right. \\ &\qquad \left. + \frac{3}{4}\sin\left(\frac{9}{16}\right)\right) \\ &\approx 0.1348234536 \end{aligned}$$

From sketch, $L_4$ underestimates $I$.

# Solutions

Let $I = \int_0^1 x \sin(x^2)\ dx$

2. Draw a sketch of $R_4$, and then calculate $R_4$.
   Based on your sketch, does $R_4$ overestimate or underestimate $I$?



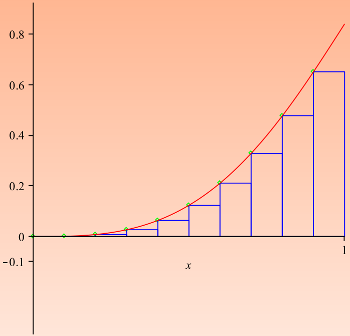

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$\begin{aligned} L_4 &= \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f(1) \\ &= \frac{1}{4}\left(\frac{1}{4}\sin\left(\frac{1}{16}\right) + \frac{2}{4}\sin\left(\frac{4}{16}\right)\right. \\ &\qquad \left. + \frac{3}{4}\sin\left(\frac{9}{16}\right) + 1\sin(1)\right) \\ &\approx 0.3451911998 \end{aligned}$$

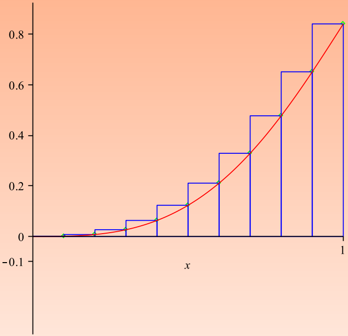
From sketch, $R_4$ underestimates $I$.

# Solutions

3. Let $I = \int_0^1 x\sin(x^2)\ dx$. Sketch $L_{10}$ and $R_{10}$.




$L_4$




$R_4$

As before, since $x\sin(x^2)$ is increasing, $L_4$ underestimates and $R_4$ overestimates $I$.

# Solutions

4. Let $I = \int_0^1 x\sin(x^2)\ dx$. Find the exact value of $I$ by using $u$-substitution. Does your result agree with your previous answers?

Let $u = x^2$. Then $du = 2x\ dx$.
Also, when $x = 0$, $u = 0^2 = 0$, and when $x = 1$, $u = 1^2 = 1$.

$$\begin{aligned} I &= \frac{1}{2}\int_0^1 \sin(x^2)2x\ dx \\ &= \frac{1}{2}\int_0^1 \sin(u)\ du = -\frac{1}{2}\cos(u)\Big|_0^1 \\ &= -\frac{1}{2}\Big(\cos(1) - \cos(0)\Big) \approx 0.2298488470 \end{aligned}$$

As expected, our result for $L_4$ under-estimated this signed area, and our result for $R_4$ over-estimated it.