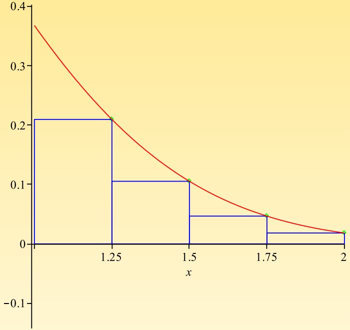
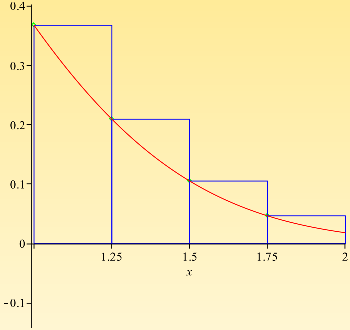
Example: Approximating $I = \int_1^2 e^{-x^2}\,dx$




0.4
0.3
0.2
0.1
0
−0.1
1.25
1.5
1.75
2
x

Partition: $1 < \frac{5}{4} < \frac{6}{4} < \frac{7}{4} < 2$

$$R_4 = \frac{1}{4}\left(f\left(\frac{5}{4}\right) + f\left(\frac{6}{4}\right) + f\left(\frac{7}{4}\right) + f(2)\right)$$

$$R_4 \approx \frac{1}{4}\left(e^{-\frac{25}{16}} + e^{-\frac{36}{16}} + e^{-\frac{49}{16}} + e^{-4}\right)$$

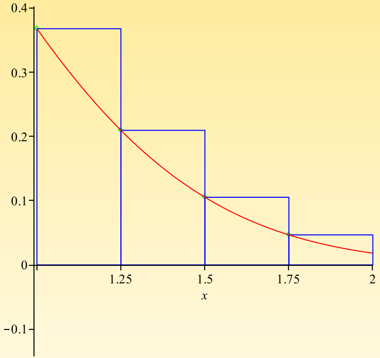
Partition: $1 < \frac{5}{4} < \frac{6}{4} < \frac{7}{4} < 2$

$$L_4 = \frac{1}{4}\left(f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{6}{4}\right) + f\left(\frac{7}{4}\right)\right)$$

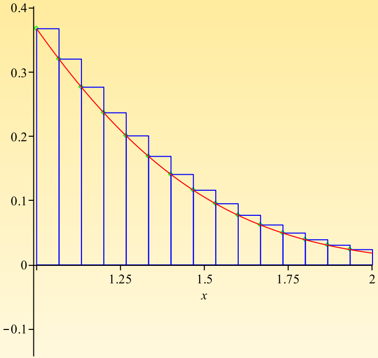
$$L_4 \approx \frac{1}{4}\left(e^{-1} + e^{-\frac{25}{16}} + e^{-\frac{36}{16}} + e^{-\frac{49}{16}}\right.$$

$$R_4 \le I \le L_4.$$

More Rectangles means a Better Approximation

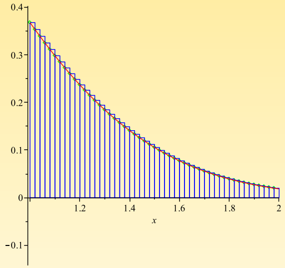


L_4

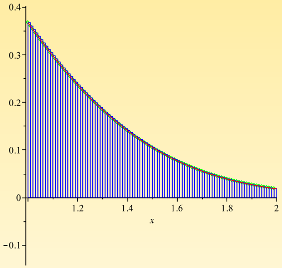


L_{15}

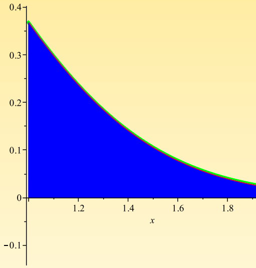
More Rectangles means a Better Approximation



L_{50}



L_{100}



L_{1000}

Leading Up To the Formal Definition of the Integral:

1. Definite integral= signed area under a curve
   Analogy: derivative = slope of the tangent line/rate of change
2. Conjectured, then proved the Fundamental Theorem of Calculus
   Analogy: showed deriv +/- where fn $\uparrow$ / $\downarrow$ early on.

3. After learning much about derivative, formally defined it:

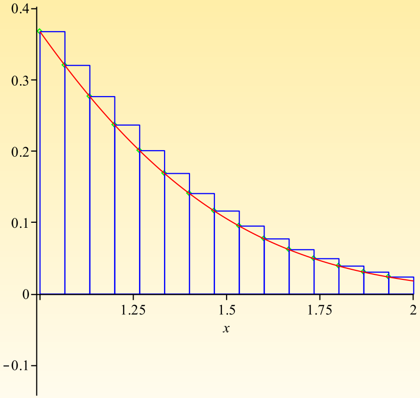
$$f'(x) \stackrel{def}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

   Similarly, will use limits to formally define of the integral.

4. As the rectangles gets narrower, left sums (or right sums, or midpoint sums) will become better and better approximations of the integral. If we take the limit as the number of rectangles approaches infinity (i.e as the width of the rectangles approaches 0), any of these will give us the exact area.

Motivating Notation:

Write L_{15} for $\int_1^2 e^{-x^2}\,dx$.

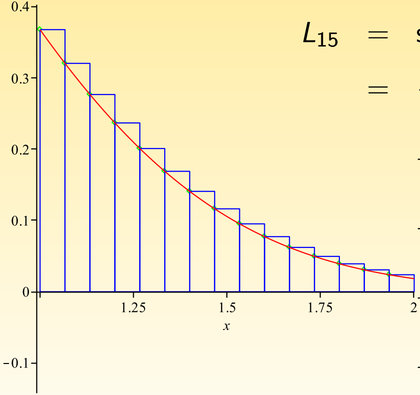


Endpoints in the partition:

$$
\begin{aligned}
x_0 &= 1\\
x_1 &= 1+\frac{1}{15}=\frac{16}{15}\\
x_2 &= 1+\frac{2}{15}=\frac{17}{15}\\
x_3 &= 1+\frac{3}{15}=\frac{18}{15}\\
&\vdots\\
x_{14} &= 1+\frac{14}{15}=\frac{29}{15}\\
x_{15} &= 2
\end{aligned}
$$

Motivating Notation:

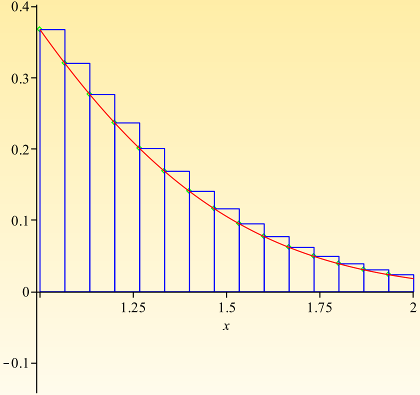
Write L_{15} for $\int_1^2 e^{-x^2}\,dx$.



$$\begin{aligned}
L_{15} &= \text{sum of areas of rectangles} \\
&= \frac{1}{15}\cdot f(1) + \frac{1}{15}\cdot f(\frac{16}{15}) + \frac{1}{15}\cdot f(\frac{17}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{18}{15}) + \frac{1}{15}\cdot f(\frac{19}{15}) + \frac{1}{15}\cdot f(\frac{20}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{21}{15}) + \frac{1}{15}\cdot f(\frac{22}{15}) + \frac{1}{15}\cdot f(\frac{23}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{24}{15}) + \frac{1}{15}\cdot f(\frac{25}{15}) + \frac{1}{15}\cdot f(\frac{16}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{27}{15}) + \frac{1}{15}\cdot f(\frac{28}{15}) + \frac{1}{15}\cdot f(\frac{29}{15})
\end{aligned}$$

Motivating Notation:

Write L_{15} for $\int_1^2 e^{-x^2}\,dx$.



YUCK!

In Class Practice

Expand the following sums currently written using sigma notation as short-hand:

1. $\displaystyle\sum_{k=0}^{4} 2k$
2. $\displaystyle\sum_{j=1}^{4} \frac{1}{4} e^{j/4}$

Write the following sum using sigma notation as short-hand:

1. $\dfrac{1}{5} + \dfrac{2}{5} + \dfrac{3}{5} + \dfrac{4}{5} + 1$
2. $\dfrac{1}{5}\cos\left(\dfrac{1}{5}\right) + \dfrac{1}{5}\cos\left(\dfrac{2}{5}\right) + \dfrac{1}{5}\cos\left(\dfrac{3}{5}\right) + \dfrac{1}{5}\cos\left(\dfrac{4}{5}\right) + \dfrac{1}{5}\cos(1)$

Solutions

Expand the following sums currently written using sigma notation as short-hand:

1. $\displaystyle\sum_{k=0}^{4} 2k = 2\cdot 0 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4$
2. $\displaystyle\sum_{j=0}^{4} \frac{1}{4} e^{j/4} = \frac{1}{4}e^{0} + \frac{1}{4}e^{1/4} + \frac{1}{4}e^{2/4} + \frac{1}{4}e^{3/4} + \frac{1}{4}e^{1}$

Write the following sum using sigma notation as short-hand:

1. $\displaystyle\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + 1 = \sum_{j=1}^{5} \frac{j}{5}$
2. $\displaystyle\frac{1}{5}\cos\left(\frac{1}{5}\right) + \frac{1}{5}\cos\left(\frac{2}{5}\right) + \frac{1}{5}\cos\left(\frac{3}{5}\right) + \frac{1}{5}\cos\left(\frac{4}{5}\right) + \frac{1}{5}\cos(1)$

   $\displaystyle = \sum_{j=1}^{5} \frac{1}{5}\cos\left(\frac{j}{5}\right)$