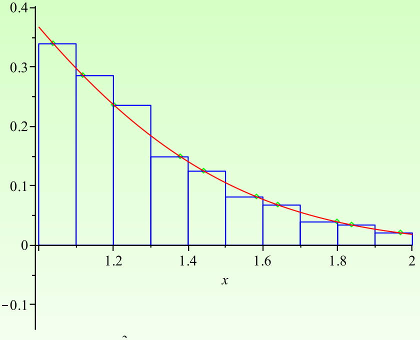
Recall:

- Any sum of areas of non-overlapping rectangles where the heights reflect the height of the function (and that don't overlap) is called a **Riemann sum**.



- Left, Right, and Midpoints sums are all types of **Reimann sum.**

Recall:

- **Definition:** Let $[a, b]$ be partitioned into n equal subintervals by $n + 1$ points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

  and let Δx be the width of the each subinterval.

  In the ith subinterval, pick a point c_i. The **Riemann sum** for f and this partition is

$$f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$$

- **Notice:** Reimann Sums give us more flexibility in how we determine the height of our rectangles. They still are just approximations of the signed area between our curve and the x-axis.

Recall:

We developed sigma notation (or summation notation) as a short-hand method of writing sums whose terms follow predictable patterns.

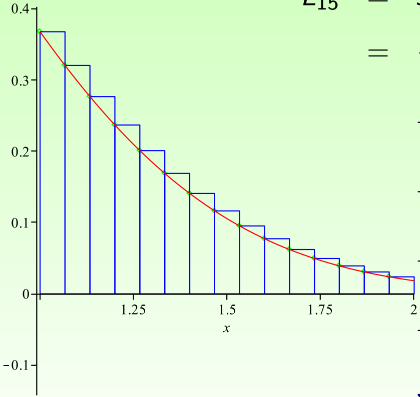
Examples:

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2$$

$$\sum_{j=0}^{3} 2j + 1 = (2 \cdot 0 + 1) + (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1)$$

Recall:

Write L_{15} for $\int_1^2 e^{-x^2}\,dx$.



$$
\begin{aligned}
L_{15} &= \text{sum of areas of rectangles} \\
&= \frac{1}{15}\cdot f(1) + \frac{1}{15}\cdot f(\frac{16}{15}) + \frac{1}{15}\cdot f(\frac{17}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{18}{15}) + \frac{1}{15}\cdot f(\frac{19}{15}) + \frac{1}{15}\cdot f(\frac{20}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{21}{15}) + \frac{1}{15}\cdot f(\frac{22}{15}) + \frac{1}{15}\cdot f(\frac{23}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{24}{15}) + \frac{1}{15}\cdot f(\frac{25}{15}) + \frac{1}{15}\cdot f(\frac{16}{15}) \\
&\quad + \frac{1}{15}\cdot f(\frac{27}{15}) + \frac{1}{15}\cdot f(\frac{28}{15}) + \frac{1}{15}\cdot f(\frac{29}{15}) \\
&= \sum_{i=0}^{14} \frac{1}{15} f\left(1 + \frac{i}{15}\right) = \sum_{i=0}^{14} \frac{1}{15} e^{-(1+i/15)^2}
\end{aligned}
$$

Recall:

Formal Definition of the Definite Integral:

$\int_a^b f(x)\,dx$ is **defined** to be the number (if one exists) to which all Riemann sums tend as the number of all subdivisions tends to ∞.

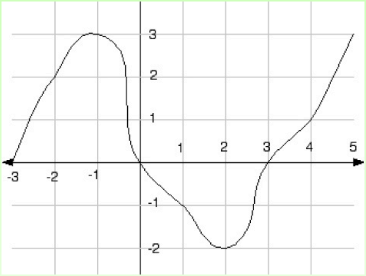
In symbols,

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\Big(\text{any Reimann sum with } n \text{ rectangles}\Big)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x\end{aligned}$$

- c_i is any x in the ith subinterval of the partition, and $f(c_i)$ can thus be thought of as the height of the ith rectangle.
- Δx is the width of all the rectangles. (Remember $\Delta x = \frac{b-a}{n}$).
- Notice how similar the two sides of this equation look. dx is the infinitesimal analogue of Δx.

In Class Work

Suppose f is the function whose graph is shown below.



Estimate the value of $\int_{-3}^{5} f(x)\,dx$ by

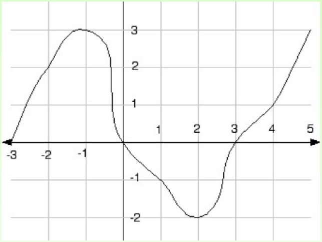
(a) evaluating a left sum with 4 equal subintervals.

(b) evaluating a right sum with 4 equal subintervals.

(c) evaluating a midpoint sum with 4 equal subintervals.

Solutions

(a) Estimate the value of $\int_{-3}^{5} f(x)\ dx$ by evaluating a left sum with 4 equal subintervals.



$$\Delta x = \frac{5-(-3)}{4} = 2$$

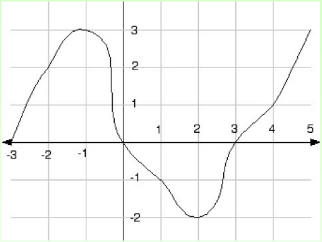
Partition: $-3 < -1 < 1 < 3 < 5$

Lefthand endpoints: -3, -1, 1, 3

$$\begin{aligned} L_4 &= 2\cdot f(-3) + 2\cdot f(-1) + 2\cdot f(1) + 2\cdot f(3) \\ &= 2\cdot 0 + 2\cdot 3 + 2\cdot(-1) + 2\cdot 0 \\ &= 4 \end{aligned}$$

Solutions

(b) Estimate the value of $\int_{-3}^{5} f(x)\ dx$ by evaluating a right sum with 4 equal subintervals.



$$\Delta x = \frac{5-(-3)}{4} = 2$$

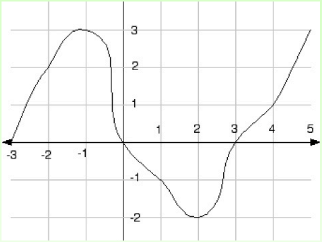
Partition: $-3 < -1 < 1 < 3 < 5$

Righthand endpoints: -1, 1, 3, 5

$$\begin{aligned} R_4 &= 2 \cdot f(-1) + 2 \cdot f(1) + 2 \cdot f(3) + 2 \cdot f(5) \\ &= 2 \cdot 3 + 2 \cdot (-1) + 2 \cdot 0 + 2 \cdot 3 \\ &= 10 \end{aligned}$$

Solutions

(c) Estimate the value of $\int_{-3}^{5} f(x)\,dx$ by evaluating a midpoint sum with 4 equal subintervals.



$$\Delta x = \frac{5-(-3)}{4} = 2$$

Partition: $-3 < -1 < 1 < 3 < 5$

The midpoints of the four subintervals:
-2, 0, 2, 4

$$\begin{aligned} M_4 &= 2\cdot f(-2) + 2\cdot f(0) + 2\cdot f(2) + 2\cdot f(4) \\ &= 2\cdot 2 + 2\cdot 0 + 2\cdot -2 + 2\cdot 1 \\ &= 2 \end{aligned}$$