Recall:

- The n-th degree Taylor Polynomial for $f(x)$ with basepoint x_0 is given by the formula

$$P_n(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

- We saw that for $f(x) = \cos(x)$ and $x_0 = 0$

$$P_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

and that for $f(x) = \sin(x)$ and $x_0 = 0$

$$P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

- Notice that we could also have found this out using $\big(\cos(x)\big)' = -\sin(x)$.

In Class Work

1. For $f(x) = \ln(x)$, $x_0 = 1$:
   (a) Find $P_5(x)$, the fifth degree Taylor polynomial for $f(x)$ based at x_0.

   (b) If you have access to graphing technology, verify your answer by graphing $P_5(x)$ and $f(x)$ on the same set of axes.

   (c) Use $P_5(x)$ to find an approximation for $\ln(1.5)$. Without using your calculator to find $\ln(1.5)$ "exactly", how can you tell whether this is larger or smaller than the exact value?

   (d) Find $P_5(1)$, $P_5'(1)$, $P_5''(1)$, $P_5'''(1)$, $P_5^{(4)}(1)$, $P_5^{(5)}(1)$ and $P_5^{(6)}(1)$, and compare the results to $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$, $f^{(4)}(1)$, $f^{(5)}(1)$ and $f^{(6)}(1)$.

   (e) What do you think $P_{15}(x)$ is?

Solutions

1. (a) Find $P_5(x)$, the fifth degree Taylor polynomial for $f(x) = \ln(x)$ and $x_0 = 1$.

$P_5(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5,$

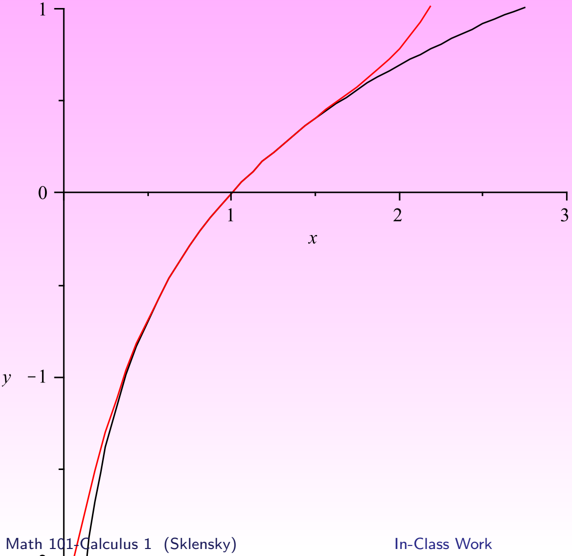
k	$f^{(k)}(x)$	$f^{(k)}(1)$	a_k
0	$\ln(x)$	0	$\frac{0}{0!} = \frac{0}{1} = 0$
1	$\frac{1}{x} = x^{-1}$	1	$\frac{1}{1!} = 1$
2	$-x^{-2} = -\frac{1}{x^2}$	-1	$\frac{-1}{2!} = -\frac{1}{2}$
3	$2x^{-3} = \frac{2}{x^3}$	2	$\frac{2}{3!} = \frac{1}{3}$
4	$-3!x^{-4} = -\frac{3!}{x^4}$	$-3!$	$-\frac{3!}{4!} = -\frac{1}{4}$
5	$4!x^{-5} = \frac{4!}{x^5}$	$4!$	$\frac{4!}{5!} = \frac{1}{5}$

Thus we have

$$P_5(x) = 0 + 1(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}.$$

Solutions

1(b). Verify your answer by graphing $P_5(x)$ and $f(x)$ on the same set of axes.

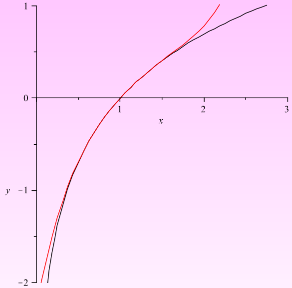


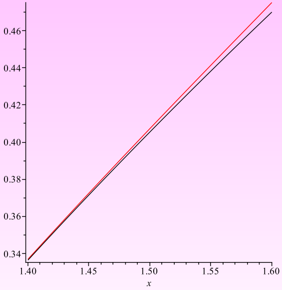
Solutions

1(c). Use $P_5(x)$ to find an approximation for $\ln(1.5)$.

$$\ln(1.5) \approx P_5(1.5) = .5 - \frac{.5^2}{2} + \frac{.5^3}{3} - \frac{.5^4}{4} + \frac{.5^5}{5} = 0.4073.$$

From a graph of $P_5(x)$ and $\ln(x)$ near $x = 1.5$, this is an over-estimate.





The estimate Maple gives for $\ln(1.5)$ is $\ln(1.5) = .4055$. Pretty close!

Solutions

$$
\begin{aligned}
\text{1(d) } P_5(x) &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} \\
P_5'(x) &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 \\
P_5''(x) &= -1 + 2(x-1) - 3(x-1)^2 + 4(x-1)^3 \\
P_5'''(x) &= 2 - 3 \cdot 2(x-1) + 4 \cdot 3(x-1)^2 \\
P_5^{(4)}(x) &= -3! + 4 \cdot 3 \cdot 2(x-1) \qquad P_5^{(5)}(x) = 4! \qquad P_5^{(6)}(x) = 0
\end{aligned}
$$

$\ln(1)$	0	$P_5(1)$	0
$\left(\frac{1}{x}\right)\Big\vert_{x=1}$	1	$P_5'(1)$	1
$\left(-\frac{1}{x^2}\right)\Big\vert_{x=1}$	-1	$P_5''(1)$	-1
$\left(\frac{2}{x^3}\right)\Big\vert_{x=1}$	2	$P_5'''(1)$	2
$\left(-\frac{3 \cdot 2}{x^4}\right)\Big\vert_{x=1}$	$-3!$	$P_5^{(4)}(1)$	$-3!$
$\left(\frac{4!}{x^5}\right)\Big\vert_{x=1}$	$4!$	$P_5^{(5)}(1)$	$4!$
$-\left(\frac{5!}{x^6}\right)\Big\vert_{x=1}$	$-5!$	$P_5^{(6)}(1)$	0

Solutions

1(e). What do you think $P_{15}(x)$ is?

Looking at the pattern, it looks to me as if

$$\begin{aligned} P_{15}(x) &= 0 + 1(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} \\ &\quad - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \ldots - \frac{(x-1)^{14}}{14} + \frac{(x-1)^{15}}{15} \end{aligned}$$

Note: If we use P_{15} to approximate $\ln(1.5)$, we get

$$\ln(1.5) \approx P_{15}(1.5) = 0.4054657568.$$

Compare to

$$\ln(1.5) = 0.4054651081...$$

Much closer, for not much extra effort!

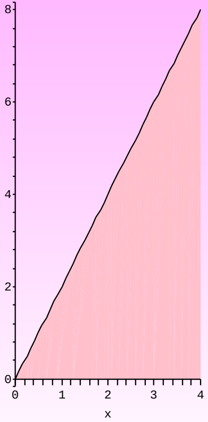
More In Class Work

Using only what we know so far – that the integral is the *signed* area between the graph and the x-axis – evaluate the following integrals by graphing the function over the designated region.

1. $\int_0^4 2x \, dx$
2. $\int_{-1}^0 2x \, dx$
3. $\int_{-1}^4 2x \, dx$
4. $\int_{-1}^1 x^3 \, dx$
5. $\int_0^\pi \cos(x) \, dx$
6. $\int_2^0 x + 2 \, dx$

Solutions

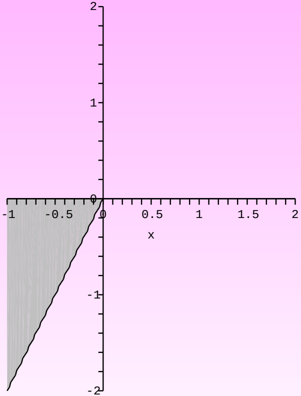
1. $\int_0^4 2x\, dx$



- All of the area is above the x-axis, so the integral will be positive.
- This is just a triangle with base 4 and height 8.
- $\int_0^4 2x\, dx = \frac{1}{2}(4)(8) = 16.$

Solutions

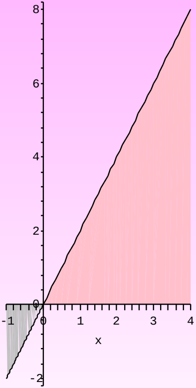
2. $\int_{-1}^{0} 2x\, dx$



- All of the area is below the x-axis, so the integral will be negative.
- This is just a triangle with base 1 and height 2.
- $\int_{-1}^{0} 2x\, dx = -\frac{1}{2}(1)(2) = -1.$

Solutions

3. $\int_{-1}^{4} 2x\, dx$



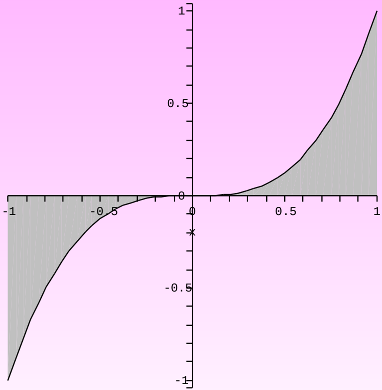
- The region from -1 to 0 is negative, and the region from 0 to 4 is positive.
- $\int_{-1}^{4} 2x\, dx = \int_{-1}^{0} 2x\, dx + \int_{0}^{4} 2x\, dx$

$$= -1 + 16$$

$$= 15.$$

Solutions

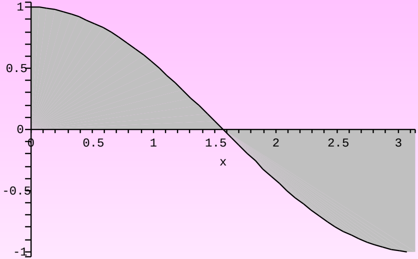
4. $\int_{-1}^{1} x^3 \, dx$



- Half of this graph is above the x-axis while the other half is below.
- The two sides are symmetric across the origin – so the negative part will cancel out the positive part.
- $\int_{-1}^{1} x^3 \, dx = 0$

Solutions

5. $\int_0^{\pi} \cos(x)\, dx$



As with the previous problem, the signed area of the region above the x-axis will cancel out with signed area of the region below the x-axis, so

$$\int_0^{\pi} \cos(x)\, dx = 0.$$

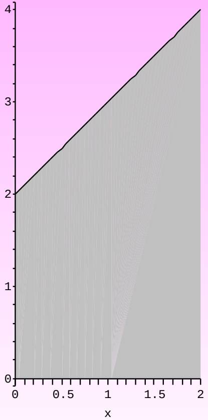
Solutions

6. $\int_2^0 x + 2 \, dx$
   - The interval goes backwards: from 2 to 0.
   - With regular area, this doesn't matter.
   - Since integrals are signed area, and since they are defined as going *from* $x = a$ *to* $x = b$, direction matters.
   - Thus when moving from right to left rather than from left to right, we expect the signed area to be different.
   - The absolute value of it (the area) can't be depend on direction, so the only thing that *can* be different is the sign.

$$\boxed{\int_2^0 x + 2 \, dx = -\int_0^2 x + 2 \, dx}$$

Solutions

6. $\int_2^0 x+2\,dx = -\int_0^2 x+2\,dx$



- This region is made up of a triangle sitting on a rectangle.
- The rectangle has $b=2$ and $h=2$ (and so is a square), with area 4.
- The triangle has $b=2$ and $h=2$, so area is $\frac{1}{2}(4)=2$.
- Thus $\int_0^2 x+2\,dx = 2+4=6$
- And hence $\int_2^0 x+2\,dx = -6$.