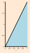
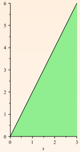
Examples:

$$\int_0^{1} 2x\,dx = \frac{1}{2}(1)(2) = 1$$

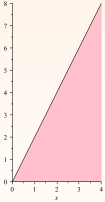


$$\int_0^{2} 2x\,dx = \frac{1}{2}(2)(4) = 4$$

$$\int_0^{3} 2x\,dx = \frac{1}{2}(3)(6) = 9$$

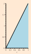


$$\int_0^{4} 2x\,dx = \frac{1}{2}(4)(8) = 16$$



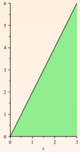
Examples:

$$\int_0^{1} 2x\,dx = \frac{1}{2}(1)(2) = 1$$

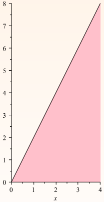


$$\int_0^{2} 2x\,dx = \frac{1}{2}(2)(4) = 4$$

$$\int_0^{3} 2x\,dx = \frac{1}{2}(3)(6) = 9$$



$$\int_0^{4} 2x\,dx = \frac{1}{2}(4)(8) = 16$$



What is $\int_0^{a} 2x\,dx$?

In Class Work

1. Let $f(x) = 2$ and $a = 0$. Then $A_f(x) = \int_0^x 2\ dt$.
   (a) By sketching a graph, find $A_f(0)$, $A_f(3)$, $A_f(-3)$, $A_f(4)$
   (b) Based on the evidence you found in part (a), you may have a hypothesis about what a formula for $A_f(x)$ might be. Show that this hypothesis is true for $x > 0$, again using a graph
   (c) Show that your hypothesis is true for $x < 0$.
   (d) How is your result, $A_f(x)$, related to the original function $f(x) = 2$?
2. Let $f(x) = 3$ and $a = 1$. Following a procedure similar to in Problem 1(b) and 1(c), find a formula for $A_f(x)$.

   HINT: You may want to break it into two situations: one where $x > 1$ and one where $x < 1$.

Solutions:

1. Let $f(x) = 2$ and $a = 0$. Then $A_f(x) = \int_0^x 2\,dt$.

   (a) Find $A_f(0)$, $A_f(3)$, $A_f(-3)$, $A_f(4)$

$$
\begin{aligned}
A_f(0) &= \int_0^0 2\,dt = \textit{signed}\text{ area from 0 to 0 under } y = 2 = \boxed{0} \\
A_f(3) &= \int_0^3 2\,dt = \textit{signed}\text{ area from 0 to 3 under } y = 2 \\
&= \text{area of rectangle of base 3 and height } 2 = \boxed{6} \\
A_f(-3) &= \int_0^{-3} 2\,dt = \textit{signed}\text{ area from 0 to -3 under } y = 2 \\
&= -(\textit{ signed}\text{ area from -3 to 0 under } y = 2) \\
&= -(\text{ area of rectangle with base 3 and height } 2) = \boxed{-6} \\
A_f(4) &= \int_0^4 \textit{signed}\text{ area from 0 to 4 under } y = 2 \\
&= \text{area of rectangle of base 4 and height } 2 = \boxed{8}
\end{aligned}
$$

Solutions:

1. Let $f(x) = 2$ and $a = 0$. Then $A_f(x) = \int_0^x 2\, dt$.

   (b) Based on the evidence you found in part (a), you may have a hypothesis about what a formula for $A_f(x)$ might be. Show that this hypothesis is true for $x > 0$.

   Summarizing our results, we find that

x	$A_f(x)$
0	0
3	6
-3	-6
4	8

   It seems as if $A_f(x) = 2x$.

   For $x > 0$, we have a rectangle of base x and height 2, so $A_f(x) = 2x$.

Solutions:

1. Let $f(x) = 2$ and $a = 0$. Then $A_f(x) = \int_0^x 2\,dt$.

   (c) Show that your hypothesis is true for $x < 0$.
   This is a little bit trickier because of the signs.
   For $x < 0$, we're going backwards.

   $$A_f(x) = \int_0^x 2\,dt = -\int_x^0 2\,dt.$$

   Also, while the height of our rectangle is still 2, the base is no longer x, as lengths must be positive. Since x is negative, the length of our base is $-x$. Thus

   $$A_f(x) = -2(-x) = 2x.$$

Solutions

2. Let $f(x) = 3$ and $a = 1$. Find a formula for $A_f(x)$.

   $A_f(x) = \int_1^x 3\ dt$= signed area from $t = 1$ to $t = x$ between the horizontal line $y = 3$ and the x-axis. No matter where x is (except if it's 1) this is a rectangle of height 3. The base depends on x.

   When $x > 1$**:** Because x is to the right of 1, and the region is above the x axis, the signed area will be positive.

   $\text{h} = 3,\ \text{b} = \text{dist from 1 to x} = x - 1 \Longrightarrow A_f(x) = 3(x-1)$.

   When $x = 1$**:** This is a region with zero area, so $A_f(1) = 0$

   When $x < 1$**:** Because x is to the left of 1, $A_f(x) = -\int_x^1 3\ dt$.

   $\text{h} = 3,\ \text{b} = \text{dist from x to } 1 = 1-x \Rightarrow A_f(x) = -3(1-x) = 3(x-1)$.

   Thus in all three cases, $\boxed{\int_1^x 3\ dt = 3(x-1) = 3x - 3.}$

In Class Work

Let $f(x) = 2x + 4$.

1. Use what we already know to find a formula for $F(x) = \int_0^x f(t)\,dt$. (That is, there is no need to draw more pictures.)
2. Use (a) and techniques from Section 5.1 to find $G(x) = \int_{-2}^x f(t)\,dt$.
3. Use (a) or (b) and techniques from Section 5.1 to find $H(x) = \int_1^x f(t)\,dt$.

Solutions

Let $f(x) = 2x + 4$.

1. Use what we already know to find a formula for $F(x) = \int_0^x f(t)\,dt$.

$$\begin{aligned}\int_0^x 2t + 4\,dt &= 2\int_0^x t\,dt + 2\int_0^x 2\,dt \\ &= 2\left(\frac{1}{2}x^2\right) + 2(2x) = x^2 + 4x\end{aligned}$$

2. Use (a) and techniques from Section 5.1 to find $G(x) = \int_{-2}^x f(t)\,dt$

$$\begin{aligned}\int_{-2}^x 2t + 4\,dt &= \int_{-2}^0 2t + 4\,dt + \int_0^x 2t + 4\,dt \\ &= \left(\text{area of a } \triangle \text{ of base 2 and height 4}\right) + \left(x^2 + 4x\right) \\ &= \left(x^2 + 4x\right) + (4)\end{aligned}$$

Solutions

Let $f(x) = 2x + 4$.

3. Use (a) or (b) and techniques from Section 5.1 to find $H(x) = \int_1^x f(t)\, dt$

$$\begin{aligned} \int_1^x 2t + 4\, dt &= \int_{-2}^x 2t + 4 - \int_{-2}^1 2t + 4\, dt \\ &= \left(x^2 + 4x + 4\right) - \left(\text{ area of triangle of base 3 and heig} \right. \\ &= \left(x^2 + 4x + 4\right) - (9) \\ &= x^2 + 4x - 5 \end{aligned}$$

Summary :

$$\begin{aligned}
\int_0^x 2\,dt &= 2x \\
\int_1^x 3\,dt &= 3x - 3 \\
\int_0^x t\,dt &= \frac{1}{2}x^2 \\
\int_0^x 2t + 4\,dt &= x^2 + 4x \\
\int_{-2}^x 2t + 4\,dt &= x^2 + 4x + 4 \\
\int_1^x 2t + 4\,dt &= x^2 + 4x - 5
\end{aligned}$$

In every example we've looked at so far, the area function is an antiderivative of the original function. Coincidence?