Guidelines when Optimizing:

- **Draw a picture** and label it. Do **not** try to visualize in your head.
- Determine what the variables are
- Decide which quantity is to be optimized
- Write an expression for the qty to be optimized–**objective function**

  (The objective is to optimize it). Often originally involves 2 or more variables. It is an expression, not an equation.

- Determine any constraints upon the variables: is there a condition that must be satisfied by the variables? Are the variables related to each other in some way?

  An equation that describes a condition the variables must satisfy is a **constraint equation**. (This is an equation, not an expression).

- If applicable, use the constraint equation to rewrite the objective function in terms of only one variable. Revisit the constraints.

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- **Draw a picture** and label it. Do **not** try to visualize in your head.
- Determine what the variables are
- Decide which quantity is to be optimized
- Write expression for the qty to be optimized: **objective function**
- If applicable, determine the constraints, incl. **constraint equation**
- If applicable, use the constraint equation to rewrite the objective function in terms of only one variable. Revisit the constraints.
- Determine the max and min values (if any) of the objective function
- Be sure to answer the question that is asked.

In Class Work

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

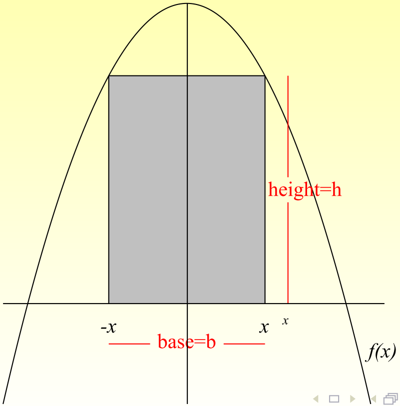
2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

   Hint: Rather than mimimizing the distance to the origin, you can minimize the *square* of the distance. This will make the algebra easier.

Solutions:

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Draw a picture:**
  To form a rectangle with both of its upper two vertices on the parabola, it must go from $-x$ to x for some value of x.



Solutions:

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Determine what the variables are:** A =area, b =base, h =height.
- **Determine which quantity is to be optimized:** Maximize area
- **Objective Function:** $A = bh$
- **Determine the constraints:**

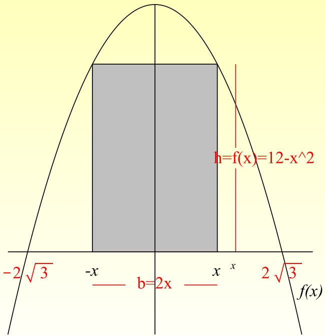
• Upper vertices lie on the parabola
$\Rightarrow$ rectangle symmetric about $x = 0$
$\Rightarrow$ Base is from $-x$ to x, so $b = 2x$.

• **Upper** 2 vertices on the parabola
$\Rightarrow$ x is btw 2 x-intercepts.

$$12 - x^2 = 0 \quad \Rightarrow \quad 12 = x^2$$
$$\Rightarrow x = \pm\sqrt{12} \quad \Rightarrow \quad x = \pm 2\sqrt{3}$$

• Height is determined by f at x, so
$h = f(x) = 12 - x^2$.

Re-label picture:



Solutions:

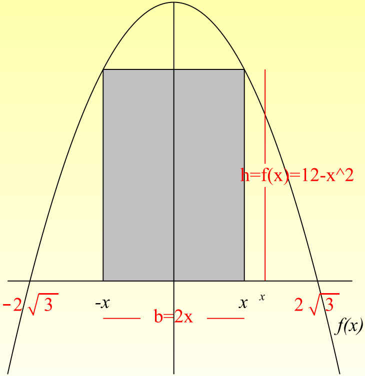
1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Rewrite the objective function in one variable:**

$$\begin{aligned} A &= bh \\ A(x) &= (2x)(12 - x^2) \\ &= 24x - 2x^3, \end{aligned}$$

with $0 \leq x \leq 2\sqrt{3}$.

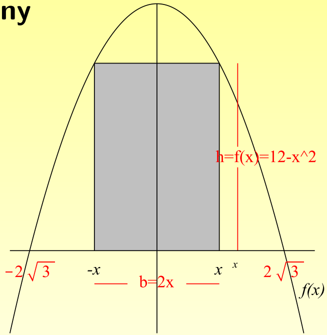
Objective: Maximize $A(x)$



Solutions:

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Determine the max and min values, if any**



$$A(x) = 24x - 2x^3,\ \text{w/ } x \in [0, 2\sqrt{3}].$$

$$A'(x) = 24 - 6x^2.$$

  - A' exists everywhere
  - $A'(x) = 0 \Rightarrow 24 = 6x^2 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2.$

$[0, 2\sqrt{3}]$ closed $\Rightarrow$ abs max & min values occur at $x = 0$, 2, or $2\sqrt{3}$.

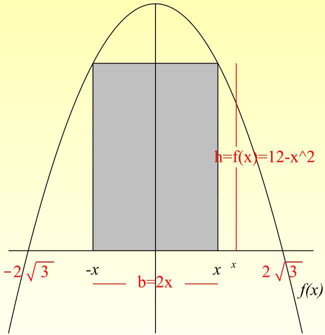
$A(0) = 24(0) - 2(0)^3 = 0,\ A(2) = 24(2) - 2(2)^3 = 32,\ A(2\sqrt{3}) = 0.$

Thus the abs max area is 32 and the abs minimum such area is 0.

Solutions:

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

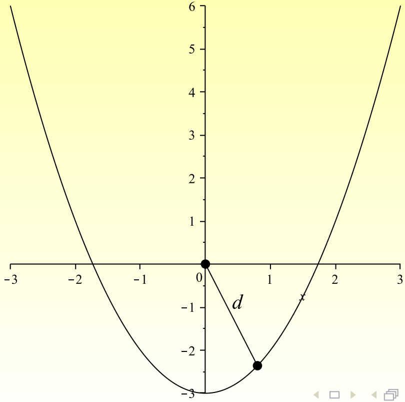
- **Answer the question:**
  The largest area such a rectangle can have is 32 square units.



Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.
(Hint: Rather than mimimizing the distance to the origin, you can minimize the *square* of the distance. This will make the algebra easier.)

- **Draw a picture**



Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.
(Hint: Rather than mimimizing the distance to the origin, you can minimize the *square* of the distance. This will make the algebra easier.)

- **Variables:**
  We have an unknown point, (x, y), and distance,
  $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
- **Quantity to be optimized:**
  Minimize distance between a point and the origin
- **Objective Function:**
  Distance from the origin $(0, 0)$ to a point (x, y):
  $d = \sqrt{x^2 + y^2}$.
  But it's easier to deal with the squaring the distance:

  Objective Function: $D = d^2 = x^2 + y^2$, minimize D.

Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

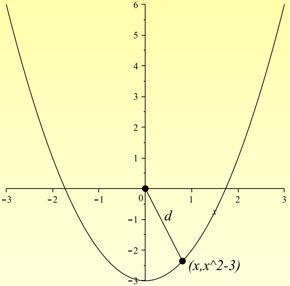
Objective Function: $D = d^2 = x^2 + y^2$, minimize D.

- **Constraints:**

  The point lies on the parabola $y = x^2 - 3$.

  This puts some constraint on us, so our **constraint equation** is $y = x^2 - 3$.

  There's no restriction on x, so we don't have endpoints.



- **Rewrite Objective Function, using Constraints:**
  Substituting the constraint $y = x^2 - 3$ into our objective function,

$$D(x) = x^2 + (x^2 - 3)^2$$

Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

Objective Function: $D = x^2 + (x^2 - 3)^2 = x^4 - 5x^2 + 9$, minimize D.

- **Find the max and min values (if any) of the objective fn:**
  - Critical numbers:
    $D'(x) = 4x^3 - 10x = 2x(2x^2 - 5)$
    $D'(x)$ exists everywhere.
    $D'(x) = 0 \Rightarrow x = 0$ or $2x^2 - 5 = 0 \Rightarrow x = 0, x = \pm\sqrt{\frac{5}{2}}$
  - Classify:
    Use the 1st or 2nd derivative test
    I'll use 2nd deriv test, since finding D'' is easy enough.

$$D'(x) = 4x^3 - 10x \Rightarrow D''(x) = 12x^2 - 10.$$

$$D''(0) = -10 < 0 \implies D \frown \text{ at } x = 0 \Rightarrow \text{local max}$$

$$D''\left(\pm\sqrt{\frac{5}{2}}\right) = 20 \implies D \smile \text{ at } x = \pm\sqrt{\frac{5}{2}} \Rightarrow \text{ local min}$$

Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

- **Answer the question:**

  There's a local max at $x = 0$ (which is by no means the abs max) and local mins (which **are** the absolute mins) at $x = \pm\sqrt{5/2}$. Thus the points on the parabola that are closest to the origin are

$$\left(-\sqrt{\frac{5}{2}}, -\frac{1}{2}\right), \left(\sqrt{\frac{5}{2}}, -\frac{1}{2}\right)$$