The Fundamental Theorem of Calculus

- **FTC, First Form:** Let f be continuous on an open interval I containing a. The function A_f defined by

$$A_f(x) = \int_a^x f(t)\,dt$$

is defined for all $x \in I$ and $\frac{d}{dx}(A_f(x)) = f(x)$. That is, A_f is an *antiderivative* of f.

  - **Consequence:** If f is continous, then f has an antiderivative, A_f. This doesn't tell us how to find it, only that it exists.

- **FTC, Second Form:** Let f be continuous on $[a, b]$, and let F be **any** antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

In Class Work

1. Evaluate the following integrals:

   (a) $\int_0^{\pi/2} \cos(x)\, dx$ (c) $\int_1^2 \frac{3}{x} - \sec^2(x)\, dx$

   (b) $\int_1^4 x^3 - \frac{2}{x^2}\, dx$ (d) $\int_0^1 x^{12} e^{x^{13}}\, dx$

2. Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\, dt$.

   (a) Find the equation of the line tangent to $y = F(x)$ at $x = 3$.

   (b) Find a formula for $\frac{d}{dx}\left(F(x^3)\right)$.

3. Find the area of the region bounded by the graphs $y = x^2$ and $y = 2x + 3$.

Solutions

1(a) $\int_0^{\pi/2} \cos(x)\, dx$

$\sin(x)$ is an antiderivative of $\cos(x)$, so from the FTC v2, we know

$$\int_0^{\pi/2} \cos(x)\, dx = \sin(x)\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1.$$

1(b) $\int_1^4 x^3 - \frac{2}{x^2}\, dx = \int_1^4 x^3 - 2x^{-2}$

$\frac{x^4}{4} - \frac{2x^{-1}}{-1}$ is an antiderivative of $x^3 - \frac{2}{x^2}$, so from the FTC v2, we know

$$\int_1^4 x^3 - \frac{2}{x^2}\, dx = (\frac{x^4}{4} + \frac{2}{x})\Big|_1^4 = (\frac{4^4}{4} + \frac{2}{4}) - (\frac{1}{4} + \frac{2}{1}) = 62 + \frac{1}{4} = \frac{249}{4}$$

Solutions

1(c) $\displaystyle\int_1^2 \frac{3}{x} - \sec^2(x)\ dx$

$3\ln(x) - \tan(x)$ is an antiderivative of $\frac{3}{x} - \sec^2(x)$, so from the FTC v2, we know

$$\int_1^2 \frac{3}{x} - \sec^2(x)\ dx = 3\ln(x) - \tan(x)\Big|_1^2 = 3\ln(2) - \tan(2) - 3\ln(1) + \tan(1)$$

Solutions

1(d) $\int_0^1 x^{12} e^{x^{13}}\, dx$

- **Need to find:** an antiderivative $F(x)$ of $x^{12}e^{x^{13}}$.
  - $e^{x^{13}}$ is a composition, and x^{12} is (more or less) $\frac{d}{dx}x^{13}$, so this came from the chain rule.
  - Chain rule: $[f(u)]' = f'(u)u'$.
  - If $f(u) = e^u$, $u = x^{13}$. $u' = 13x^{12}$, then $f'(u)u' = 13x^{12}e^{x^{13}}$ – not quite what we have.
  - If $f(u) = \frac{1}{13}e^u$, $u = x^{13}$, $u' = 13x^{12}$, then $f'(u)u' = 13x^{12}\frac{1}{13}e^{x^{13}} = x^{12}e^{x^{13}}$–WHAT WE HAVE!
  - $F(x) = \frac{1}{13}e^{x^{13}}$
- **Using FTC v2,**

$$\int_0^1 x^{12}e^{x^{13}}\, dx = \frac{1}{13}e^{x^{13}}\Big|_0^1 = \frac{1}{13}e^{1^{13}} - \frac{1}{13}e^{0^{13}} = \frac{1}{13}(e-1).$$

Solutions

2. Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\,dt$.

   (a) Find the equation of the line tangent to $y = F(x)$ at $x = 3$.
   Need to find: Slope of line and point on line

   - **Point on the tangent line:** $(3, F(3))$

$$F(3) = \int_1^3 f(t)\,dt = \int_1^3 2t\cos(t^2)\,dt.$$

Antiderivative of $2t\cos(t^2)$:
• $\cos(t^2)$ is a composition, and $2t$ is the derivative of t^2, so this came from the Chain Rule.
• $\dfrac{df(u)}{dx} = f'(u)u'(t)$. If $f(u) = \sin(u)$ and $u(t) = t^2$, then $f'(u)u'(t) = \cos(u)\cdot 2t = 2t\cos(t^2)$ - what we have.
• Thus an antiderivative of $2t\cos(t^2)$ is $\sin(t^2)$.
Thus

$$F(3) = \sin(t^2)\Big|_1^3 = \sin(9) - \sin(1) \approx -0.43.$$

Point on the line: $(3, -0.43)$

Solutions

2. Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\,dt$.

   (a) Find the equation of the line tangent to $y = F(x)$ at $x = 3$.
   Need to find: Slope of line and point on line

   - **Point on the tangent line:** $(3, -0.43)$
   - **Slope of the tangent line:** $F'(3)$
     To find $F'(x)$, use FTC, v1:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x),$$

so $F'(x) = f(x) = 2x\cos(x^2)$, and thus

$$F'(3) = 2(3)\cos(9) = 6\cos(9) \approx -5.47.$$

   (b) **Equation of the tangent line:**

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - (\sin(9) - \sin(1)) &= 6\cos(9)(x-3) \\ y + 0.43 &\approx -5.47(x-3) \end{aligned}$$

Solutions:

2. Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\, dt$.

   (b) Find a formula for $\frac{d}{dx}\left(F(x^3)\right)$.
   Let $G(x) = F(x^3) = F(u)$, where $u(x) = x^3$.
   From the chain rule:

   $$G'(x) = F'(u)u'(x) = F'(u) \cdot 3x^2.$$

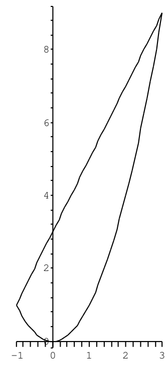
   From the FTC, v1, $F'(x) = f(x) = 2x\cos(x^2)$, so
   $F'(u) = 2u\cos(u^2) = 2x^3\cos(x^6)$.
   Therefore

   $$\frac{d}{dx}(F(x^3)) = [2x^3\cos(x^6)] \cdot (3x^2) = 6x^5\cos(x^6).$$

   Note: In this particular case, because we are able to antidifferentiate $2x\cos(x^2)$, we could have found $F(x^3)$ and then differentiated, but (once you get used to it) this is faster.

Solutions:

3. Find the area of the region bounded by the graphs $y = x^2$ and $y = 2x + 3$.



The area bounded by the two graphs is what we get when we start with the area under the line $y = 2x + 3$, and take away the area under the parabola $y = x^2$.

$$A = \int_{\text{left int. pt}}^{\text{right int. pt}} 2x + 3 \, dx - \int_{\text{left int. pt}}^{\text{right int. pt}} x^2 \, dx$$

$$= \int_{\text{left int. pt}}^{\text{right int. pt}} 2x + 3 - x^2 \, dx$$

Need to find the left and right intersection points:

$x^2 = 2x+3 \Rightarrow x^2-2x-3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3$ or $x = -1$

Thus $A = \displaystyle\int_{-1}^{3} 2x + 3 - x^2 \, dx$

Solutions:

3. Find the area of the region bounded by the graphs $y = x^2$ and $y = 2x + 3$.

$$A = \int_{-1}^{3} 2x + 3 - x^2 \, dx$$

$$\begin{aligned}
A &= \left[x^2 + 3x - \frac{1}{3}x^3\right]\Big|_{-1}^{3} \\
&= [(3)^2 + 3(3) - \frac{1}{3}(3)^3] - [(-1)^2 + 3(-1) - \frac{1}{3}(-1)^3] \\
&= [9 + 9 - 9] - [1 - 3 - \frac{1}{3}(-1)] \\
&= 9 - [-2 + \frac{1}{3}] \\
&= 9 - (-5/3) \\
&= 32/3
\end{aligned}$$