The Fundamental Theorem of Calculus

- **FTC, First Form:** Let f be continuous on an open interval I containing a. The function A_f defined by

$$A_f(x) = \int_a^x f(t)\, dt$$

is defined for all $x \in I$ and $\frac{d}{dx}(A_f(x)) = f(x)$. That is, A_f is an *antiderivative* of f.

  - **Consequence:** If f is continous, then f has an antiderivative, A_f. This doesn't tell us how to find it, only that it exists.

- **FTC, Second Form:** Let f be continuous on $[a, b]$, and let F be **any** antiderivative of f. Then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Substitution

1. Observe: have a product; one factor is a composition.
   Plan: replace all terms involving x and dx w/ equivalent terms involving u and du to create a simpler (less cluttered) integrand.
2. Choose u to be the inside of the composition. Use u to find du.
3. Substitute in u and du.
   Note: du is not exactly a variable. In an integral, can only have du. Can't have $(du)^2$, $\frac{1}{du}$, $\sqrt{du}$.

   > If you are able to completely eliminate all mention of x and dx, while still only having one du, then your substitution was successful.
   >
   > Whether it was useful remains to be seen.

4. Antidifferentiate your simpler integrand in terms of u.
   If you are able to do so, your successful substitution was also useful!
5. Replace u with its equivalent expression in terms of x.

In Class Work

Use substitution to find the following indefinite integrals, and *check your results*.

1. $\displaystyle\int \cos(x)e^{\sin(x)}\,dx \qquad (u = \sin(x))$
2. $\displaystyle\int x\sin(\pi x^2)\,dx \qquad (u = \pi x^2)$
3. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1-x)$

Solutions:

1. $\int \cos(x)e^{\sin(x)}\,dx \qquad (u = \sin(x))$

   - Integrand= product; one factor =composition. Came from chain rule?
   - Choose u to be the inside function in the composition (if there is one).
     Composition: $e^{\sin(x)}$, Inside function: $u = \sin(x)$.
   - Find du:
     Since $u = \sin(x)$, then $\frac{du}{dx} = \cos(x)$, so $du = \cos(x)\,dx$.
   - Substitute in u and du, without changing integral
     $$\int \cos(x)e^{\sin(x)}\,dx = \int e^u\,du$$
   - Antidifferentiate
     $$\int \cos(x)e^{\sin(x)}\,dx = \int e^u\,du = e^u + C$$
   - Substitute back in for x
     $$\int \cos(x)e^{\sin(x)}\,dx = e^{\sin(x)} + C$$

Solutions:

2. $\int x\sin(\pi x^2)\,dx \qquad (u=\pi x^2)$

- Integrand= product; one factor =composition. Came from chain rule?
- Choose u to be the inside function in the composition (if there is one). Composition: $\sin(\pi x^2)$, Inside function: $u=\pi x^2$.
- Find du:
  Since $u=\pi x^2$, then $\dfrac{du}{dx}=2\pi x$, so $du=2\pi\,dx$.
- Substitute in u and du, without changing integral

$$\int x\sin(\pi x^2)\,dx=\frac{1}{2\pi}\int \sin(\pi x^2)(2\pi x\,dx)=\frac{1}{2\pi}\int \sin(u)\,du$$

- Antidifferentiate

$$\int x\sin(\pi x^2)\,dx=\frac{1}{2\pi}\int \sin(u)\,du=\frac{1}{2\pi}\big(-\cos(u)\big)+C$$

- Substitute back in for x

$$\int x\sin(\pi x^2)\,dx=-\frac{1}{2\pi}\cos(\pi x^2)+C$$

Solutions:

3. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1-x)$

- Integrand= product; one factor =composition. Came from chain rule?
- Choose u to be the inside function in the composition (if there is one).
  Composition: $\sqrt{1-x} = (1-x)^{1/2}$, Inside function: $u = 1-x$.
- Find du:
  Since $u = 1-x$, then $\frac{du}{dx} = -1$, so $du = -1\,dx$.
- Substitute in u and du, without changing integral

$$\int \frac{1}{\sqrt{1-x}}\,dx = -1\int (1-x)^{-1/2}(-1\,dx) = -1\int u^{-1/2}\,du$$

- Antidifferentiate

$$\int \frac{1}{\sqrt{1-x}}\,dx = -1\int u^{-1/2}\,du = (-1)(2)u^{1/2} + C$$

- Substitute back in for x

$$\int \frac{1}{\sqrt{1-x}}\,dx = -2\sqrt{1-x} + C$$