Guidelines when Optimizing:

- **Draw a picture** and label it. Do **not** try to visualize in your head.
- Determine what the variables are
- Decide which quantity is to be optimized
- Write an expression for the qty to be optimized–**objective function**

  (The objective is to optimize it). Often originally involves 2 or more variables. It is an expression, not an equation.

- Determine any constraints upon the variables: is there a condition that must be satisfied by the variables? Are the variables related to each other in some way?

  An equation that describes a condition the variables must satisfy is a **constraint equation**. (This is an equation, not an expression).

- If applicable, use the constraint equation to rewrite the objective function in terms of only one variable. Revisit the constraints.

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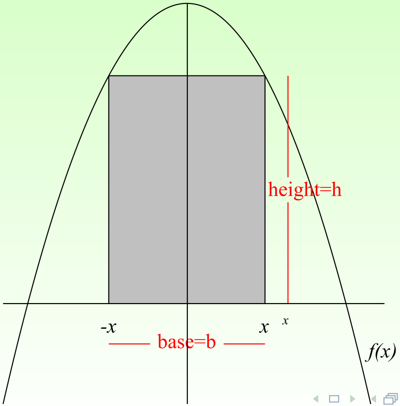
- **Draw a picture** and label it. Do **not** try to visualize in your head.
- Determine what the variables are
- Decide which quantity is to be optimized
- Write expression for the qty to be optimized: **objective function**
- If applicable, determine the constraints, incl. **constraint equation**
- If applicable, use the constraint equation to rewrite the objective function in terms of only one variable. Revisit the constraints.
- Determine the max and min values (if any) of the objective function
- Be sure to answer the question that is asked.

Solutions:

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Draw a picture:**
  To form a rectangle with both of its upper two vertices on the parabola, it must go from $-x$ to x for some value of x.



Solutions:

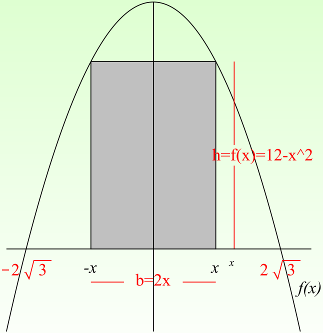
1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Determine what the variables are:** $A =$area, $b =$base, $h =$height.
- **Determine which quantity is to be optimized:** Maximize area
- **Objective Function:** $A = bh$
- **Determine the constraints:**
  - Upper vertices lie on the parabola $\Rightarrow$ rectangle symmetric about $x = 0$ $\Rightarrow$ Base is from $-x$ to x, so $b = 2x$.
  - **Upper** 2 vertices on the parabola $\Rightarrow$ x is btw 2 x-intercepts.

$$12 - x^2 = 0 \quad \Rightarrow \quad 12 = x^2$$
$$\Rightarrow x = \pm\sqrt{12} \quad \Rightarrow \quad x = \pm 2\sqrt{3}$$

  - Height is determined by f at x, so $h = f(x) = 12 - x^2$.

Re-label picture:



Solutions:

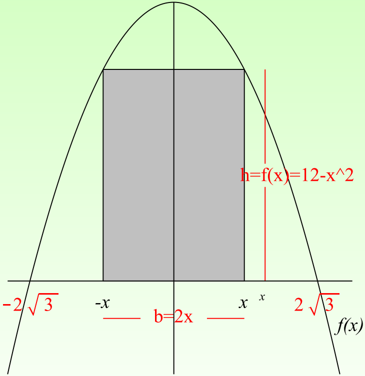
1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Rewrite the objective function in one variable:**

$$\begin{aligned} A &= bh \\ A(x) &= (2x)(12 - x^2) \\ &= 24x - 2x^3, \end{aligned}$$

with $0 \leq x \leq 2\sqrt{3}$.

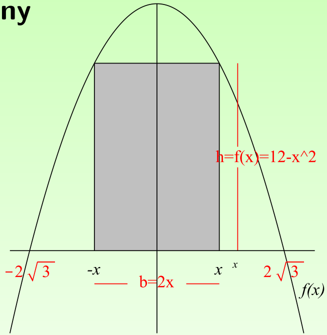
Objective: Maximize $A(x)$



Solutions:

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

- **Determine the max and min values, if any**



$$A(x) = 24x - 2x^3, \text{ w/ } x \in [0, 2\sqrt{3}].$$

$$A'(x) = 24 - 6x^2.$$

- A' exists everywhere
- $A'(x) = 0 \Rightarrow 24 = 6x^2 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2.$

$[0, 2\sqrt{3}]$ closed $\Rightarrow$ abs max & min values occur at $x = 0$, 2, or $2\sqrt{3}$.

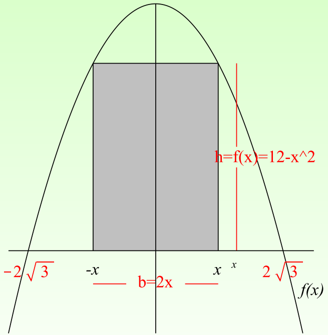
$$A(0) = 24(0) - 2(0)^3 = 0, \ A(2) = 24(2) - 2(2)^3 = 32, \ A(2\sqrt{3}) = 0.$$

Thus the abs max area is 32 and the abs minimum such area is 0.

Solutions:

1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

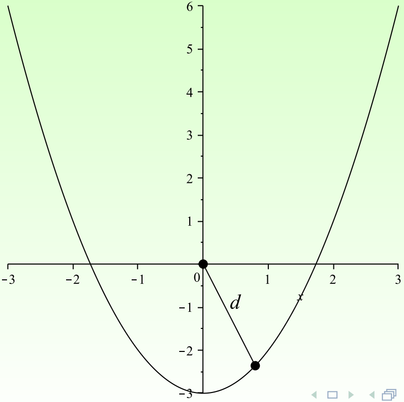
- **Answer the question:**
  The largest area such a rectangle can have is 32 square units.



Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.
(Hint: Rather than mimimizing the distance to the origin, you can minimize the *square* of the distance. This will make the algebra easier.)

- **Draw a picture**



Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.
(Hint: Rather than mimimizing the distance to the origin, you can minimize the *square* of the distance. This will make the algebra easier.)

- **Variables:**
  We have an unknown point, (x, y), and distance,
  $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
- **Quantity to be optimized:**
  Minimize distance between a point and the origin
- **Objective Function:**
  Distance from the origin $(0, 0)$ to a point (x, y):
  $d = \sqrt{x^2 + y^2}$.
  But it's easier to deal with the squaring the distance:

  Objective Function: $D = d^2 = x^2 + y^2$, minimize D.

Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

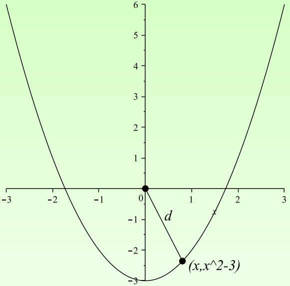
Objective Function: $D = d^2 = x^2 + y^2$, minimize D.

- **Constraints:**

  The point lies on the parabola $y = x^2 - 3$.

  This puts some constraint on us, so our **constraint equation** is $y = x^2 - 3$.

  There's no restriction on x, so we don't have endpoints.



- **Rewrite Objective Function, using Constraints:**
  Substituting the constraint $y = x^2 - 3$ into our objective function,

$$D(x) = x^2 + (x^2 - 3)^2$$

Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

Objective Function: $D = x^2 + (x^2 - 3)^2 = x^4 - 5x^2 + 9$, minimize D.

- **Find the max and min values (if any) of the objective fn:**
  - Critical numbers:
    $D'(x) = 4x^3 - 10x = 2x(2x^2 - 5)$
    $D'(x)$ exists everywhere.
    $D'(x) = 0 \Rightarrow x = 0$ or $2x^2 - 5 = 0 \Rightarrow x = 0, x = \pm\sqrt{\frac{5}{2}}$
  - Classify:
    Use the 1st or 2nd derivative test
    I'll use 2nd deriv test, since finding D'' is easy enough.

$$D'(x) = 4x^3 - 10x \Rightarrow D''(x) = 12x^2 - 10.$$

$$D''(0) = -10 < 0 \quad \Longrightarrow \quad D \frown \text{ at } x = 0 \Rightarrow \text{local max}$$

$$D''\left(\pm\sqrt{\frac{5}{2}}\right) = 20 \quad \Longrightarrow \quad D \smile \text{ at } x = \pm\sqrt{\frac{5}{2}} \Rightarrow \text{ local min}$$

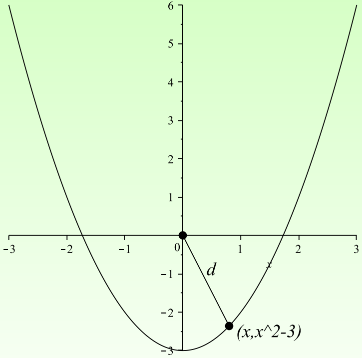
Solutions:

2. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

- **Answer the question:**

  There's a local max at $x = 0$ (which is by no means the abs max) and local mins (which **are** the absolute mins) at $x = \pm\sqrt{5/2}$. Thus the points on the parabola that are closest to the origin are

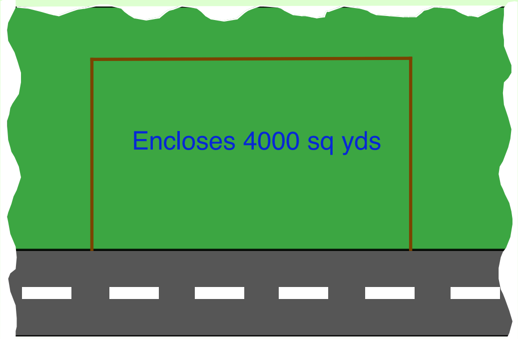
$$\left(-\sqrt{\frac{5}{2}}, -\frac{1}{2}\right), \left(\sqrt{\frac{5}{2}}, -\frac{1}{2}\right)$$



Solutions:

3. A city is planning to build a park along a major road. The park is to be rectangular with an area of 4000 square yards and will be fenced off on the three sides that are not adjacent to the road. How long and wide should the park be to minimize the amount of fencing used? That is, what is the least amount of fence required for this job?

- **Draw Picture:**



Solutions:

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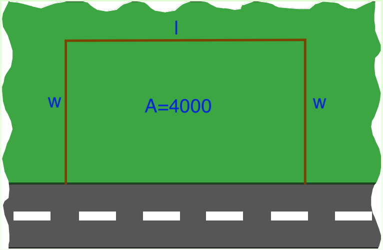
- **Variables:**

  A =area
  w =width of park
  l =length of park
  F = amount of fencing used.

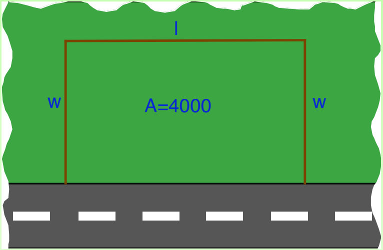


- **What is being optimized?** Minimize F.

Solutions:

A =area
w =width of park
l =length of park
F = amount of fencing used.

Minimize F



- **Objective function:** $F = l + 2w$
- **Constraints:** $F,\ l,\ w \geq 0$, with no upper limits.

  Enclose an area of 4000 yd^2 $\Rightarrow A = 4000 \Rightarrow lw = 4000$

$$\Rightarrow l = \frac{4000}{w} \quad \text{(Constraint Equation)}$$

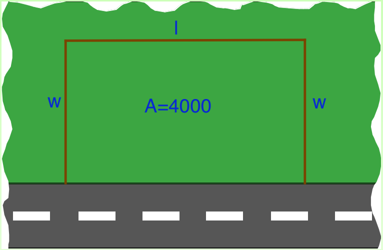
- **Use constraint to rewrite objective fn:**

$$l = \frac{4000}{w} \Rightarrow F(w) = \frac{4000}{w} + 2w.$$

Solutions:

A =area
w =width of park
l =length of park
F = amount of fencing used.

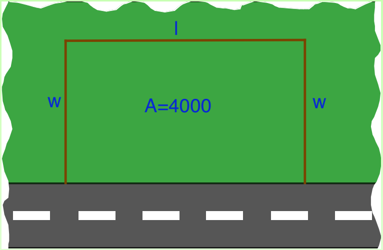
Minimize:
$$F(w) = \frac{4000}{w} + 2w$$



- **Determine Max and Min Values**
  Plan: w is not constrained to a closed interval, so find critical numbers, and then use the 1st or 2nd derivative test to determine whether F has a local minimum, local maximum, or neither at each critical number.

Solutions:

A =area
w =width of park
l =length of park
F = amount of fencing used.



Minimize:
$$F(w) = \frac{4000}{w} + 2w$$

- **Determine Max and Min Values**, continued
  - **Critical Numbers**

$$F(w) = 4000w^{-1}+2w \Rightarrow F'(w) = -4000w^{-2}+2 \Rightarrow F'(w) = 2-\frac{4000}{w^2}$$

$F'(w)$ doesn't exist at $w = 0$, but neither does F.

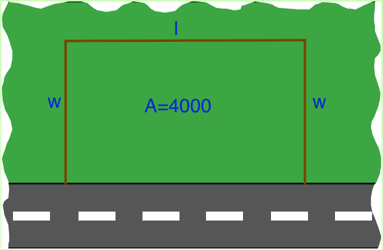
$$2-\frac{4000}{w^2} = 0 \Rightarrow 2w^2 = 4000 \Rightarrow w^2 = 2000 \Rightarrow w = \pm\sqrt{2000} = \pm 20\sqrt{5}$$

Since w must be positive, our only critical point is $w = 20\sqrt{5}$.

Solutions:

A =area
w =width of park
l =length of park
F = amount of fencing used.

Minimize:
$$F(w) = \frac{4000}{w} + 2w$$



- **Determine Max and Min Values**, continued
  - **Classify Critical Points:** $w = 20\sqrt{5}$
    Second Derivative Test:
    $$F''(w) = 8000w^3 \Longrightarrow F''(20\sqrt{5}) > 0 \Longrightarrow F \text{ concave up}$$
    Thus F has a local minimum at $20\sqrt{5}$.

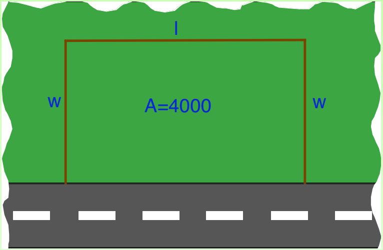
Solutions:

A =area

w =width of park

l =length of park

F = amount of fencing used.



Minimize:

$$F(w) = \frac{4000}{w} + 2w$$

- **Answer the question:** $w = 20\sqrt{5} \approx 44.72$ yards.

$$l = \frac{4000}{w} = \frac{4000}{20\sqrt{5}} = \frac{200}{\sqrt{5}} \approx 89.44 \text{ yards.}$$

The dimensions that will minimize the amount of fencing used are roughly 44.72 yds ($\perp$ to the road) $\times$ 89.44 yds ($\|$ to the road). The amount of fencing to make a park with these dimensions is

$$F(20\sqrt{5}) = \frac{4000}{20\sqrt{5}} + 2(20\sqrt{5}) \approx 89.44 + 89.44 = 178.88 \text{ yards.}$$

Solutions:

4. Group tickets to a concert are priced at \$40 per ticket if 20 tickets are ordered, but cost \$1 per ticket less for every extra ticket ordered, up to a maximum of 50 tickets. (For example, if 22 tickets are ordered, the price is \$38 per ticket.) Find the number of tickets (i.e. the size of group) that maximizes the total cost of the tickets for the group.

- **Draw a picture**: N/A
- **Variables:** $n =$number of people over 20, $t =$price per ticket, $C =$ total cost.
- **Which quantity is to be optimized?** Maximize total cost
- **Objective function:** $C = (20 + n)t$
- **Constraints:** $0 \leq n \leq 30$, $t = 40 - n$ (Constraint Equation)
- **Use constraint equation to rewrite the objective function in terms of only one variable:**

$$C(n) = (20 + n)(40 - n), 0 \leq n \leq 30$$

Solutions:

4. Maximize $C(n) = (20 + n)(40 - n)$, $0 \leq n \leq 30$

- **Optimize:**
  Because n is on a closed interval, we find the critical points, then evaluate C at each critical point and at endpoints.
  - **Critical points:**
    - $C(n) = (20 + n)(40 - n) = 800 + 20n - n^2 \Rightarrow C'(n) = 20 - 2n.$
    - $C'(n)$ exists everywhere.
    - $C'(n) = 0 \Rightarrow 20 - 2n = 0 \Rightarrow n = 10.$

    Thus the only critical point is $n = 10$ tickets.
  - Test $C(n)$ at $n = 0$, $n = 10$, and $n = 30$.

$$\begin{aligned} C(0) &= (20 + 0)(40 - 0) = 800 \\ C(10) &= (20 + 10)(40 - 10) = 900 \\ C(30) &= (20 + 30)(40 - 30) = 500 \end{aligned}$$

    The the absolute maximum of C is \$900 at $n = 10$ and the absolute minimum is \$500 at $n = 30$

Solutions:

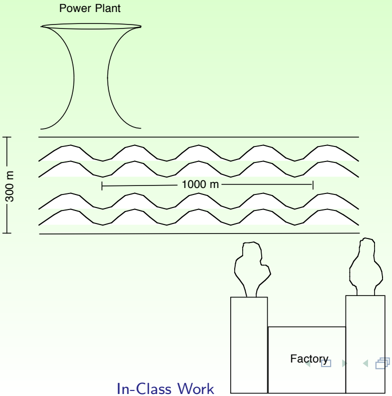
4. Suppose that group tickets to a concert are priced at \$40 per ticket if 20 tickets are ordered, but cost \$1 per ticket less for every extra ticket ordered, up to a maximum of 50 tickets. (For example, if 22 tickets are ordered, the price is \$38 per ticket.) Find the number of tickets that maximizes the total cost of the tickets.

- **Be sure to answer the question that is asked.**
  The number of tickets that maximizes the total cost of the tickets is $t = 20 + 10 = 30$ tickets, and the total cost of those 30 tickets is \$900.

Solutions:

5. A cable is to be run from a power plant on one side of a river to a factory on the other side. It costs \$4 per meter to run the cable over land, while it costs \$5 per meter to run the cable under water. Suppose the river is 300 meters wide and the factory is 1000 meters downstream from the power plant. What is the most economical route to lay the cable? How much will it cost?

- **Draw Picture:**



Solutions:

5. A cable is to be run from a power plant on one side of a river to a factory on the other side. It costs \$4 per meter to run the cable over land, while it costs \$5 per meter to run the cable under water. Suppose the river is 300 meters wide and the factory is 1000 meters downstream from the power plant. What is the most economical route to lay the cable? How much will it cost?

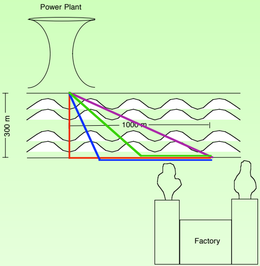
- **Variables:** C =cost of cable (in dollars), d_l =length of cable over land (in meters), d_w =length of cable under water (in meters).
- **What's being optimized?** Minimize cost
- **Objective Function:**

$$\begin{aligned} \text{total cost} &= \text{cost of cable over land} + \text{cost of cable under water} \\ C &= \text{price per m, land} d_l + \text{price per m, water} d_w \\ &= 4d_l + 5d_w \end{aligned}$$

Solutions:

- **Constraints:**
  The cable could be laid along alot of different routes.

  d_l =cable length, land
  d_w =cable length, H_2O
  C =total cost

  Minimize:
  $C = 4d_l + 5d_2$

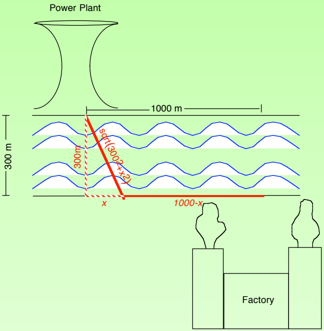
  The cable could be laid three basic ways:
  1. Straight across river, then along river's edge, $d_w = 300$, $d_l = 1000$
  2. Crossing river at angle, then along river's edge. $0 < d_l < 1000$, $300 < d_w < \sqrt{300^2 + 1000^2}$
  3. Directly from power plant to factory, w/ no time on land. $d_w = \sqrt{300^2 + 1000^2}$, $d_l = 0$.

  The first and last scenarios are our endpoints, and will lead to specific costs. The middle scenario will lead to varying costs.

Solutions:

- **Constraints:**

  d_l =cable length, land
  d_w =cable length, H_2O
  C =total cost

  Minimize:
  $C = 4d_l + 5d_2$

  If x is the distance downstream from the entry point that the cable emerges from the water, then

$$d_l = 1000 - x \qquad d_w = \sqrt{300^2 + x^2}$$

- **Use contraint equation to rewrite objective function in terms of only one variable:**

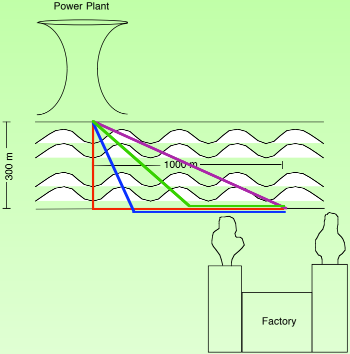
$$\begin{aligned} C(x) &= 4d_l + 5d_w = 4(1000 - x\ ) + 5(\sqrt{300^2 + x^2}) \\ &= (4000 - 4x) + 5\sqrt{300^2 + x^2} \text{ dollars} \end{aligned}$$

Solutions:

Minimize:

$C(x) = 4000 - 4x + 5\sqrt{300^2 + x^2}$

$0 \leq x \leq 1000.$



$$\begin{aligned} C'(x) &= -4 + 5 \cdot \frac{1}{2}(300^2 + x^2)^{-1/2}(2x) \\ &= \frac{5x}{\sqrt{300^2 + x^2}} - 4 \end{aligned}$$

- **Find the max and min values**

  Find the abs max & abs min by checking endpts and any critical #s

  - **Find Critical numbers of $C(x)$:**

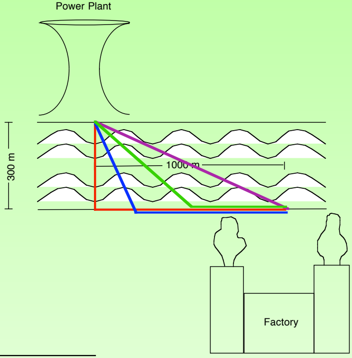
    If $x = 0$, then we're in scenario 1; if $x = 1000$, we're in scenario 3.

    - $C'(x)$ exists everywhere, since $300^2 + x^2 \geq 300^2$
    - $C'(x) = 0 \Rightarrow \dfrac{5x}{\sqrt{300^2 + x^2}} - 4 = 0 \Rightarrow 5x = 4\sqrt{300^2 + x^2}$

      $\Rightarrow \dfrac{5}{4}x = \sqrt{300^2 + x^2} \Rightarrow \dfrac{25}{16}x^2 = 300^2 + x^2 \Rightarrow \dfrac{9}{16}x^2 = 300^2$

      $\Rightarrow \dfrac{3}{4}x = \pm 300 \Rightarrow x = \pm 400$

Solutions:



Optimize:

$$C(x) = 4000 - 4x + 5\sqrt{300^2 + x^2}$$

Endpoints: $0 \le x \le 1000$

Critical Numbers: $x = \pm 400$

- **Find the max and min values**

$$\begin{aligned} C(0) &= 4000 + 5(300) = 5500 \\ C(400) &= 4000 - 4 \cdot 400 + 5\sqrt{300^2 + 400^2} \\ &= 4000 - 1600 + 5 \cdot 500 = 4900 \\ C(1000) &= 4000 - 4000 + 5\sqrt{300^2 + 1000^2} \approx 5(1044.03) = 5220.15 \end{aligned}$$

Abs max cost: \$5550, if $x = 0$, abs min cost: \$4900, if $x = 400$.

- **Answer the question:**
  Put cable in the water right outside the power plant; lay it in a straight line under water so it emerges 400 feet downstream; then lay it along river's edge for the last 600 feet. The cost will be \$4900.