Reminder: what are polynomials?

- **Examples:**
  - $3x^7 - 42x^6 + 7x^3 - x + 3$
  - $42(x-3)^3 + 7(x-2)^2 - 4$
- Any expression with just addition, subtraction, and multiplication, and only non-negative whole-number powers of x.
- **Coefficients:** The constants in front of the x's, or the constant at the end

Arbitrary polynomials:

- We write an arbitrary polynomial with *basepoint* x_0 as follows:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

  where $a_0, a_1, a_2, \ldots$ are coefficients.

- The a_i are coefficients, so they are constant.
- x_0 is a fixed basepoint, so it is constant also.
- The only variable is x

Arbitrary polynomials

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

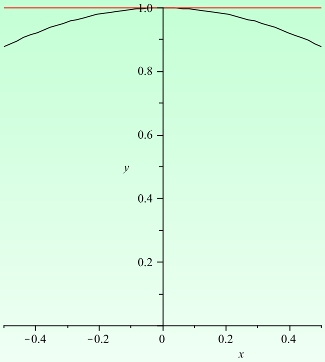
Examples:

- $P_3(x) = 1 - 2(x - 7) + \frac{1}{2}(x - 7)^2 - \pi(x - 7)^3$
  - Basepoint: $x_0 = 7$
  - Degree: $n = 3$
  - Coefficients: $a_0 = 1$, $a_1 = -2$, $a_2 = \frac{1}{2}$, and $a_3 = \pi$
- $P_4(x) = (x - \pi) + 7(x - \pi)^3 + 10(x - \pi)^4$
  - Basepoint: $x_0 = \pi$
  - Degree: $n = 4$
  - Coefficients: $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = 7$, $a_4 = 10$.

Notice that $P_3(7) = 1 = a_0$, and $P_4(\pi) = 0 = a_0$. This is part of why we call x_0 the basepoint - it's easiest to calculate the polynomial there.

Approximating complicated fns with simpler ones:

Using the tangent to approximate $\cos(x)$:



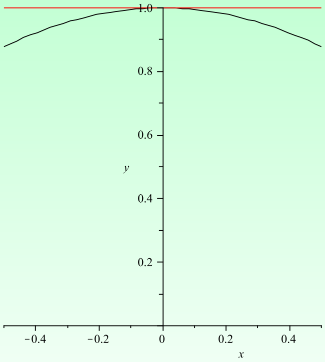
At left is the graph of $\cos(x)$ (in black) and its tangent line at $x = 0$ (in red).

Very near to the point of tangency, the tangent line gives a good approximation to the function.

Why?

Approximating complicated fns with simpler ones:

Using the tangent to approximate cos(x):



At left is the graph of $\cos(x)$ (in black) and its tangent line at $x = 0$ (in red).

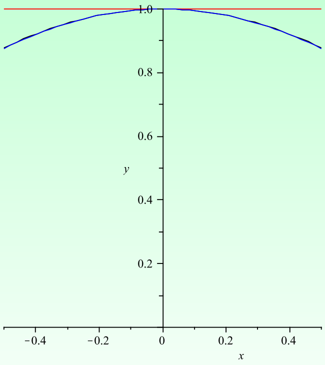
Very near to the point of tangency, the tangent line gives a good approximation to the function.

Why? Because they have the same slope and y-value at $x = 0$.

In other words, because both the functions and their first derivatives match $x = 0$.

Approximating complicated fns with simpler ones:

What if we make more derivatives agree?



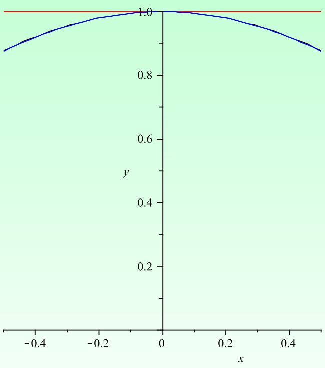
At left is the graph of $\cos(x)$ (in black), its tangent line at $x = 0$ (in red), and a new polynomial P_2 (in blue), created so that at $x = 0$, $P_2(x)$ and $\cos(x)$ not only have the same y-value and the same slope, as in the last slide, but also the same concavity.

P_2 gives such a good approximation of $\cos(x)$ over this small interval, we can't even see the difference.

Why?

Approximating complicated fns with simpler ones:

What if we make more derivatives agree?



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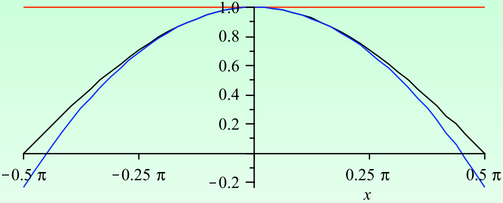
P_2 gives such a good approximation of $\cos(x)$ over this small interval, we can't even see the difference.

Why? Because its y-value, first and second derivative at $x = 0$ match $\cos(x)$.

Approximating complicated fns with simpler ones:

What if we make more derivatives agree?

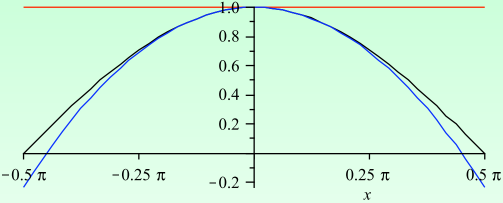


But if we look over a larger interval, we see that despite the y-value, slope, and concavity all matching $\cos(x)$ at $x = 0$, $P_2(x)$ doesn't do as good a job of approximating $\cos(x)$ if we look farther away from $x = 0$.

How can we get a still better approximation?

Approximating complicated fns with simpler ones:

What if we make more derivatives agree?



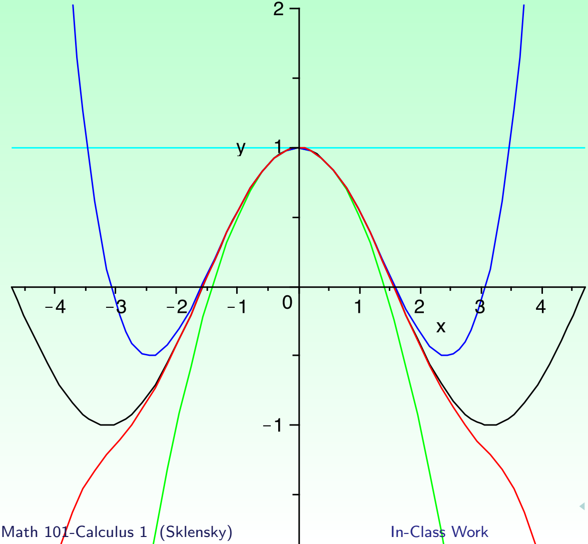
But if we look over a larger interval, we see that despite the y-value, slope, and concavity all matching $\cos(x)$ at $x = 0$, $P_2(x)$ doesn't do as good a job of approximating $\cos(x)$ if we look farther away from $x = 0$.

How can we get a still better approximation?

Try matching still more derivatives at $x = 0$

Approximating complicated fns with simpler ones:

What if we make more derivatives agree?



The more derivatives a polynomial and our function agree on at that one point, $x = 0$, the better job that polynomial does at approximating the function! (Our original function, $\cos(x)$, is in black).

Recall:

Let $P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$ be an arbitrary polynomial based at $x = x_0$.

- *Notation:* For any integer $n > 0$, $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$.
  Also $0! = 1$.

  Examples: $4! = 4 \cdot 3 \cdot 2$, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$
- What are the derivatives of $P_n(x)$ at $x = x_0$?
  - $P_n^0(x_0) = a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)^2 + \cdots + a_n(x_0 - x_0)^n = a_0$
  - $P_n'(x_0) = a_1 + 2a_2(x_0 - x_0) + 3a_3(x_0 - x_0)^2 + \cdots + na_n(x_0 - x_0)^{n-1} = a_1$
  - $P_n''(x_0) = 2a_2 + 3 \cdot 2a_3(x_0 - x_0) + \cdots + n(n-1)(x_0 - x_0)^{n-2} = 2a_2$
  - $P_n'''(x_0) = 3!a_3 + 4 \cdot 3 \cdot 2(x_0 - x_0) + \cdots + n(n-1)(n-2)(x_0 - x_0)^{n-3} = 3!a_3$

    $\vdots$
  - $P_n^{(n)}(x_0) = n!a_n$
- In general, for the kth derivative, $P_n^{(k)}(x_0) = k!a_k$.

In Class Work

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

1. Find $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$.

2. If you have a graphing calculator, verify your answer by graphing the polynomials and $f(x)$ on the same set of axes.

3. Use $P_5(x)$ to find an approximation for $\sin(3)$.

   Will this be larger or smaller than the actual value of $\sin(3)$?

4. Now find $P_{19}(x)$.
   Hint: You don't actually need to take all of the derivatives.

Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

Taylor polynomials: $P_k(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k, \;\; a_k = \dfrac{f^{(k)}(0)}{k!}$.

1. Find $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$.

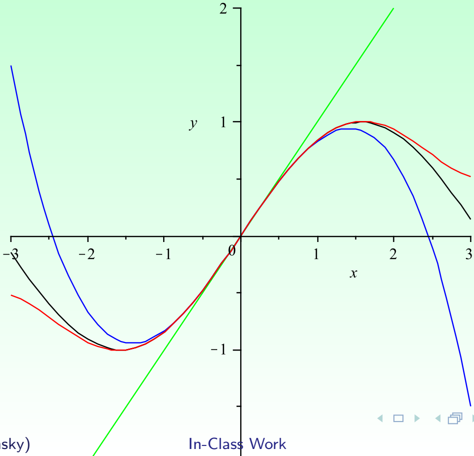
k	$f^{(k)}(x)$	$f^{(k)}(0)$	a_k
0	$\sin(x)$	0	$\frac{0}{0!} = \frac{0}{1} = 0$
1	$\cos(x)$	1	$\frac{1}{1!} = 1$
2	$-\sin(x)$	0	$\frac{0}{2!} = 0$
3	$-\cos(x)$	-1	$-\frac{1}{3!}$
4	$\sin(x)$	0	0
5	$\cos(x)$	1	$\frac{1}{5!}$

$$\begin{aligned}
P_1(x) &= 0 + 1x = x \\
P_2(x) &= 0 + 1x + 0x^2 = x \\
P_3(x) &= 0 + 1x + 0x^2 - x^3/3! \\
&= x - x^3/3! \\
P_4(x) &= 0 + 1x + 0x^2 - x^3/3! + 0x^4 \\
&= x - x^3/3! \\
P_5(x) &= 1x - x^3/3! + 0x^4 + x^5/5! \\
&= x - x^3/3! + x^5/5!
\end{aligned}$$

Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

2. Verify your answer by graphing the polynomials and $f(x)$ on the same set of axes.



Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

3. Use $P_5(x)$ to find an approximation for $\sin(3)$.
Will this be larger or smaller than the actual value of $\sin(3)$?

$$\sin(3) \approx P_5(3) = 3 - \frac{3^3}{3!} + \frac{3^5}{5!} \approx .525.$$

Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

4. Now find $P_{19}(x)$.
Hint: You don't actually need to take all of the derivatives.
It looks to me like all the even derivatives are going to be 0, and the odd ones will be ± 1, so we'll have

$$P_{19}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!} - \frac{x^{19}}{19!}.$$