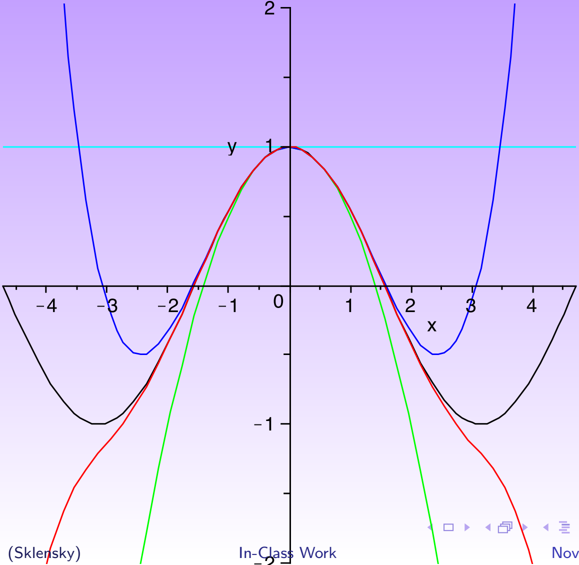
Recall:

- Suppose $f(x)$ is a function that seems impossible to evaluate exactly at most points (like $\cos(x)$, $\sin(x)$, e^x, $\ln(x)$, for instance).
- But suppose we *can* evaluate $f(x)$, and all its derivatives, at the point x_0.
- Then we can create a Taylor polynomial for $f(x)$ based at $x = x_0$ to approximate values of $f(x)$ near $x = x_0$.
- The formula for the nth degree Taylor polynomial for $f(x)$ based at $x = x_0$ is:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$P_0(x)$, $P_2(x)$, $P_4(x)$ and $P_6(x)$ for $\cos(x)$ based at $x_0 = 0$



In Class Work

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

1. Find $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$.

2. If you have a graphing calculator, verify your answer by graphing the polynomials and $f(x)$ on the same set of axes.

3. Use $P_5(x)$ to find an approximation for $\sin(3)$.

4. Now find $P_{19}(x)$.
   Hint: You don't actually need to take all of the derivatives. Look for patterns

Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

Taylor polynomials: $P_k(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k, \;\; a_k = \dfrac{f^{(k)}(0)}{k!}$.

1. Find $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ and $P_5(x)$.

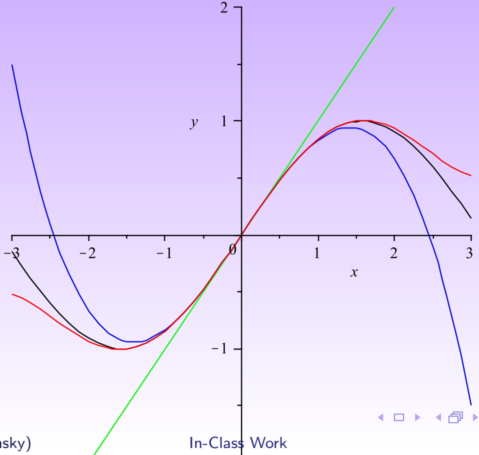
k	$f^{(k)}(x)$	$f^{(k)}(0)$	a_k
0	$\sin(x)$	0	$\frac{0}{0!} = \frac{0}{1} = 0$
1	$\cos(x)$	1	$\frac{1}{1!} = 1$
2	$-\sin(x)$	0	$\frac{0}{2!} = 0$
3	$-\cos(x)$	-1	$-\frac{1}{3!}$
4	$\sin(x)$	0	0
5	$\cos(x)$	1	$\frac{1}{5!}$

$$
\begin{aligned}
P_1(x) &= 0 + 1x = x \\
P_2(x) &= 0 + 1x + 0x^2 = x \\
P_3(x) &= 0 + 1x + 0x^2 - x^3/3! \\
&= x - x^3/3! \\
P_4(x) &= 0 + 1x + 0x^2 - x^3/3! + 0x^4 \\
&= x - x^3/3! \\
P_5(x) &= 1x - x^3/3! + 0x^4 + x^5/5! \\
&= x - x^3/3! + x^5/5!
\end{aligned}
$$

Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

2. Verify your answer by graphing the polynomials and $f(x)$ on the same set of axes.



Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

3. Use $P_5(x)$ to find an approximation for $\sin(3)$.
Will this be larger or smaller than the actual value of $\sin(3)$?

$$\sin(3) \approx P_5(3) = 3 - \frac{3^3}{3!} + \frac{3^5}{5!} \approx .525.$$

Solutions

Let $f(x) = \sin(x)$ and
let $P_k(x)$ be the kth order Taylor polynomial for $f(x)$ at $x_0 = 0$.

4. Now find $P_{19}(x)$.
Hint: You don't actually need to take all of the derivatives.
Because the derivatives cycle between four options, all the even derivatives are going to be 0, and the odd ones will be alternate between 1 and -1, so we'll have

$$P_{19}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!} - \frac{x^{19}}{19!}.$$