Summary of Wednesday's Conclusions:

- **Definition:** A number c in the domain of a function f is called a **critical number** (or a **critical point**) of f if $f'(c) = 0$ or $f'(c)$ is undefined.

- **Theorem:** Suppose that $f(c)$ is a local extremum (that is, is either a local min or a local max). Then c **must** be a critical number.

- *If* $f(x)$ has a local extremum at $x = c$, then f' **must** be 0 or undefined, but just because f' is zero or undefined doesn't *guarantee* that there's a local extremum at that point.

In Class Work

Find all critical numbers. Use graphing technology to determine whether each is a local minimum, local maximum, or neither.

1. $f(x) = x^4 - 3x^3 + 2$
2. $f(x) = \sin(x)\cos(x)$ on $[0, 2\pi]$
3. $f(x) = x^{1/2}(x^2 - 4)^3$

Solutions:

1. $f(x) = x^4 - 3x^3 + 2$

Need to find: where $f'(x) = 0$ and where $f'(x)$ is undefined.

$$f'(x) = 4x^3 - 9x^2.$$

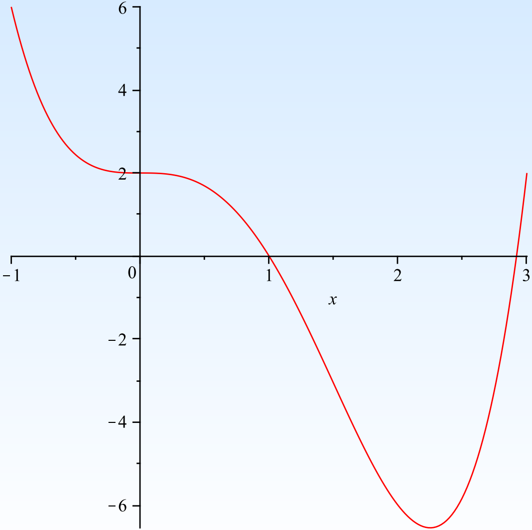
$f'(x)$ is a polynomial and so exists everywhere.

$$\begin{aligned} f'(x) = 0 \quad &\Longrightarrow \quad 4x^3 - 9x^2 = 0 \\ &\Longrightarrow \quad x^2(4x - 9) = 0 \\ &\Longrightarrow \quad x = 0 \text{ or } x = 9/4 \end{aligned}$$

Thus the only critical numbers are $x = 0$ and $x = 9/4$. That is, the only places $f(x)$ can *possibly* have a local max or a local min is at $x = 0$ or at $x = 9/4$.

Solutions

1. (continued)



Solutions

2. $f(x) = \sin(x)\cos(x)$ on $[0, 2\pi]$

Need to find: where $f'(x) = 0$ and where $f'(x)$ is undefined

$$f'(x) = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x).$$

$f'(x)$ is defined everywhere.

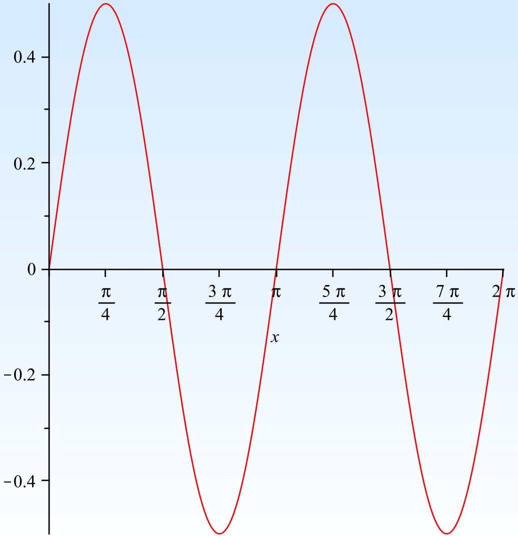
Where does $f'(x) = 0$?

$$\begin{aligned} f'(x) = 0 &\implies \cos^2(x) - \sin^2(x) \\ &\implies \cos^2(x) = \sin^2(x) \\ &\implies \cos(x) = \pm\sin(x) \\ &\implies x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4} \end{aligned}$$

Thus the critical numbers on $[0, 2\pi]$ are $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{7\pi}{4}$, and so the only points on the interval $[0, 2\pi]$ where $f(x) = \sin(x)\cos(x)$ could possibly have a local max or a local min are one of these four points.

Solutions

2. (continued)



3. $f(x) = x^{1/2}(x^2-4)^3$
Need to find: where $f'(x) = 0$ and where $f'(x)$ is undefined

$$\begin{aligned} f'(x) &= x^{1/2}\left(3(x^2-4)^2(2x)\right) + \left(\frac{1}{2}x^{-1/2}\right)(x^2-4)^3 \\ &= 6x^{3/2}(x^2-4)^2 + \frac{1}{2x^{1/2}}(x^2-4)^3 \\ &= \frac{1}{2x^{1/2}}(x^2-4)^2\left(12x^2+(x^2-4)\right) \\ &= \frac{1}{2x^{1/2}}(x^2-4)^2(13x^2-4) \end{aligned}$$

$f'(x)$ is undefined at $x = 0$ (but $f(x)$ *is* defined there).
$f'(x) = 0$ at $x = \pm 2$ and at $x = \pm\frac{2}{\sqrt{13}}$
However: f is not defined for negative values of x.

Thus the critical numbers – the only places f can possibly hope to achieve a local max or local min – are at $x = 0$, $x = 2$, $x = \frac{2}{\sqrt{13}}$

3. (continued)