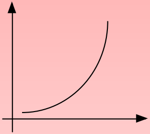
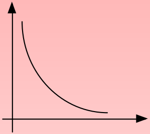
Recall

Concave up:

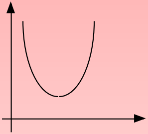
f increases at increasing rate



f decreases at increasing rate



f changes at an increasing rate

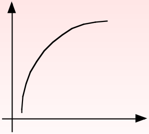


In all cases:

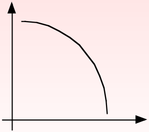
f concave up
$\Leftrightarrow f'$ increasing
$\Leftrightarrow f'' > 0$

Concave down:

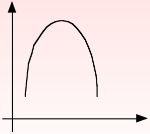
f increases at decreasing rate



f decreases at decreasing rate



f changes at a decreasing rate



In all cases:

f concave down
$\Leftrightarrow f'$ decreasing
$\Leftrightarrow f'' < 0$

Also recall:

Thus f can only change concavity where $f''(c) = 0$ or where $f''(c)$ does not exist.

In Class Work

1. Determine the intervals where the graph of $f(x) = x^2 + \frac{1}{x}$ is concave up and concave down, and find all inflection points.

Solutions

1. Determine the intervals where the graph $f(x) = x^2 + \frac{1}{x}$ is concave up and concave down, and find all inflection points.

$f(x)$ is defined for all x except for $x = 0$.

f is concave up $\Leftrightarrow f'' > 0$ $\qquad$ f is concave down $\Leftrightarrow f'' < 0$

$$f'(x) = 2x - x^{-2} \qquad \Rightarrow \qquad f''(x) = 2 + 2x^{-3} = 2 + \frac{2}{x^3}.$$

- f'' is undefined at $x = 0$ (which isn't in the domain of f)
- $f''(x) = 0 \Leftrightarrow 2 + \frac{2}{x^3} = 0 \Leftrightarrow \frac{2}{x^3} = -2 \Leftrightarrow x^3 = -1 \Leftrightarrow x = -1.$

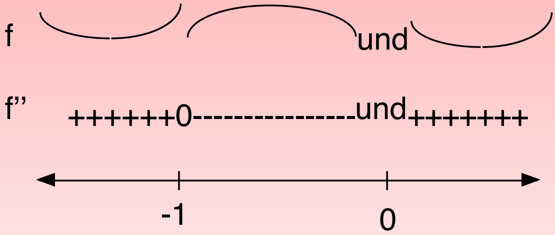
Thus f could change concavity at $x = -1$ and/or at $x = 0$. Note: since f isn't defined at $x = 0$, a change of concavity would not mean an inflection point there.

(to be continued...)

Solutions (continued)

1. $f''(x) = 2 + \dfrac{2}{x^3}$ $\quad$ $f''(x) = 0$ at $x = -1$ $\quad$ $f''(x)$ d.n.e. at $x = 0$.

$$f''(-2) = 2 + \frac{2}{(-2)^3} > 0,\ f''(-\tfrac{1}{2}) = 2 + \frac{2}{(-\frac{1}{2})^3} < 0,\ f''(1) = 2 + \frac{2}{(1)^3} > 0$$



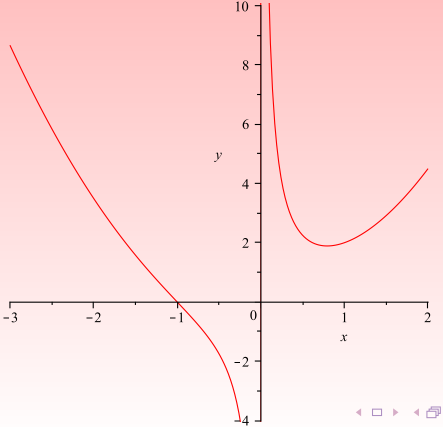
That is, f is concave up on $(-\infty, -1) \cup (0, \infty)$, and f is concave down on $(-1, 0)$.

Although f changes concavity at both $x = -1$ and at $x = 0$, only $x = 0$ is an inflection point, because f is not defined at $x = 0$.

Solutions (continued)

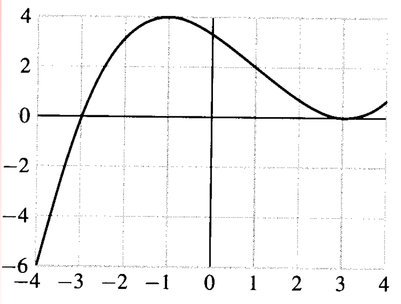
1. Found: f is conc. up on $(-\infty, -1) \cup (0, \infty)$, & f is conc. down on $(-1, 0)$.

Although f changes concavity at both $x = -1$ and at $x = 0$, only $x = 0$ is an inflection point, because f is not defined at $x = 0$.



In Class Work

2. Find all critical numbers and use the Second Derivative Test to determine all local extrema:
   (a) $f(x) = xe^{-x}$
   (b) $f(x) = \frac{x^2-5x+4}{x}$
3. Suppose $g(3) = 1$, $g'(3) = 2$, and $g''(3) = -2$. Does $g(x)$ have a local maximum at $x = 3$?
4. Let f be the function shown below:



Rank the values of $f''(-3)$, $f''(1)$, and $f''(3)$ in increasing order.

Solutions:

2. Find all critical numbers and use the Second Derivative Test to determine all local extrema:

(a) $f(x) = xe^{-x}$

- **Note:** $f(x) = xe^{-x} = \frac{x}{e^x}$ is defined for all x.
- **Critical Numbers:**

$$f'(x) = -xe^{-x} + e^{-x} = \frac{1}{e^x}\big(-x+1\big) = \frac{1}{e^x}(1-x)$$

  Thus the only critical number is $x = 1$

- **Local Extrema:**

$$f''(x) = e^{-x}(-1) - e^{-x}(1-x) = e^{-x}\big(-1-1+x\big) = e^{-x}(x-2)$$

  $f''(1) = \frac{1-2}{e^1} < 0$, so f is concave down at $x = 1$.
  Since f is flat and concave down at $x = 1$, f has a local maximum at $x = 1$.

Solutions:

2. Find all critical numbers, use the 2nd Deriv Test

(b)$f(x) = \frac{x^2-5x+4}{x}$

- **Note:** $f(x) = \frac{x^2-5x+4}{x}$ is undefined at $x = 0$.
- **Critical Numbers:**

$$f'(x) = \frac{x(2x-5)-(x^2-5x+4)(1)}{x^2} = \frac{x^2-4}{x^2} = \frac{(x-2)(x+2)}{x^2}$$

  Only critical numbers are $x = \pm 2$; $x = 0$ is not a critical number b/c f isn't defined there (**but** f could still change direction there).

- **Local Extrema:**

$$f''(x) = \frac{x^2(2x)-(x^2-4)(2x)}{x^4} = \frac{8x}{x^4} = \frac{8}{x^3}$$

$$f''(-2) = \frac{8}{(-2)^3} < 0 \quad \Rightarrow \quad f \text{ is conc down} \Rightarrow \text{local max at} x = -2$$

$$f''(2) = \frac{8}{(2)^3} > 0 \quad \Rightarrow \quad f \text{ is conc up} \Rightarrow \text{local min at } x = 2$$

Solutions:

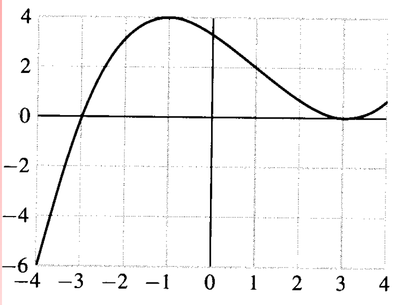
3. Suppose $g(3) = 1$, $g'(3) = 2$, and $g''(3) = -2$. Does $g(x)$ have a local maximum at $x = 3$?

Just because $g''(3) < 0$ and thus g is concave down at $x = 2$, this alone does not say whether g has a local maximum at $x = 3$. We would also need for g' to be 0 or to not exist at $x = 3$.

Since $g'(3) = 2$, $x = 3$ is not a critical number of g, and so g does not have a local maximum (or a local minimum) at $x = 3$

Solutions:

4. Let f be the function shown below:



Rank the values of $f''(-3)$, $f''(1)$, and $f''(3)$ in increasing order.

- f is concave down at $x = -3$ and hence $f''(-3) < 0$.
- f has an inflection point at $x = 1$ and hence $f''(1) = 0$
- f is concave up at $x = 3$ and hence $f''(3) > 0$.

Thus $f''(-3) < f''(0) < f''(3)$