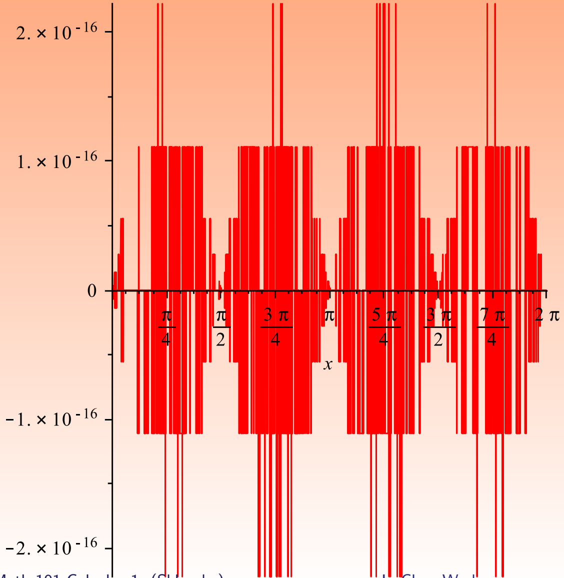
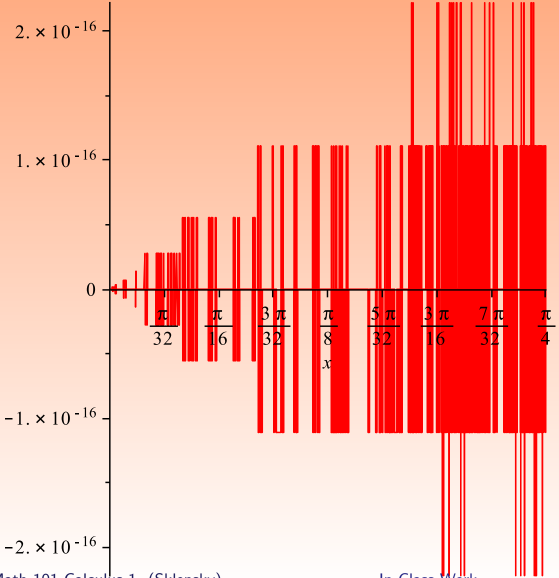
Problems with Graphing Technology

Consider this graph:



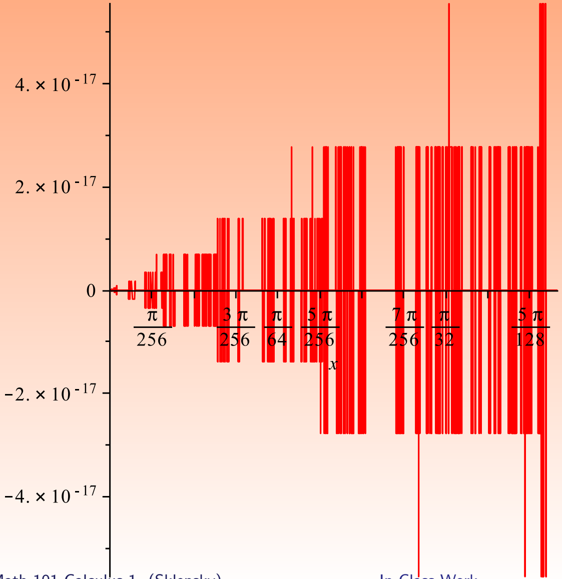
Problems with Graphing Technology

Zooming in to a domain one quarter as large,



Problems with Graphing Technology

And zooming in to an even smaller domain,



Things to investigate When Graphing a Function

- **Critical Numbers:** Find the critical numbers of f
- **Increasing/Decreasing:** On each interval determined by the critical numbers *and* the points not in the domain of f, find if f is $\uparrow$ or $\downarrow$
- **Local Extrema:** Conclude where there are local mins and max's
- **Potential inflection points:** Find critical numbers of f'
- **Concavity**: On intervals to each side of points where $f''(x) = 0$ or d.n.e., find whether f is $\smile$ or $\frown$. If the concavity changes at a point where f exists, f has an infl pt there: if f' d.n.e, it's an infl pt w/ a vert tangent, if $f'(x) = 0$, then it's an infl pt w/ a hor tangent.

continued...

Things to investigate When Graphing a Function

- **Domain** - are there any points or intervals *not* in the domain of f?
- **Vertical asymptotes** - If a is an isolated point not in the domain (not an interval), is there a vertical asymptote at a, or is it a removable discontinuity? Find $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$
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continued...

More things to investigate

- **Horizontal Asymptotes**: Check the limit of $f(x)$ as $x \to \pm\infty$. If either limit is finite, you have a horizontal asymptote on that side. Polynomials and roots do not have horizontal asymptotes.
  A function is most likely to have a horizontal asymptote if it involves an exponential function or is a quotient of two functions.

- **A few key points:** Find the y-values of all the points you've identified as important. Find the y-intercept, and if it's not too painful, find the x-intercepts as well.

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

1. **Find the domain of** $f(x)$

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

1. **Find the domain of $f(x)$**

$$f(x) = \frac{2(x^2 - 4)}{(x-1)(x+1)} = \frac{2(x-2)(x+2)}{(x-1)(x+1)}$$

The only numbers we're not allowed to plug into $f(x)$ are $x = -1$ and $x = 1$.

Domain: All $x \neq \pm 1$.

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

2. **Vertical asymptotes** - If a is an isolated point not in the domain (not an interval), is there a vertical asymptote at a, or is it a removable discontinuity? Find $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

3. **Critical Numbers:** Find the critical numbers of f
4. **Increasing/Decreasing:** On each interval determined by the critical numbers *and* the points not in the domain of f, find if f is $\uparrow$ or $\downarrow$
5. **Local Extrema:** Conclude where there are local mins and max's
6. **Potential inflection points:** Find critical numbers of f'
7. **Concavity**: On intervals to each side of points where $f''(x) = 0$ or d.n.e., find whether f is $\smile$ or $\frown$. If the concavity changes at a point where f exists, f has an infl pt there: if f' d.n.e, it's an infl pt w/ a vert tangent, if $f'(x) = 0$, then it's an infl pt w/ a hor tangent.

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

3. **Critical Numbers:** Find the critical numbers of f

$$\begin{aligned} f'(x) &= \frac{(x^2-1)(4x) - (2x^2-8)(2x)}{(x^2-1)^2} \\ &= \frac{4x^3 - 4x - 4x^3 + 16x}{(x^2-1)^2} \\ &= \frac{12x}{(x^2-1)^2} \end{aligned}$$

Thus $f'(x)$ dne at ± 1 (but neither does f), and $f'(x) = 0$ where $12x = 0$, i.e. where $x = 0$.

The function may change direction at any of these three points, but $x = 0$ is the only critical number (because f is not defined at the other two points)

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

4. **Increasing/Decreasing:** On each interval determined by the critical numbers *and* the points not in the domain of f, find if f is $\uparrow$ or $\downarrow$
5. **Local Extrema:** Conclude where there are local mins and max's
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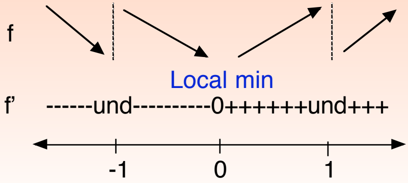
Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

4. **Increasing/Decreasing:** On each interval determined by the critical numbers *and* the points not in the domain of f, find if f is $\uparrow$ or $\downarrow$
5. **Local Extrema:** Conclude where there are local mins and max's

$$f'(x) = \frac{12x}{(x^2 - 1)^2} \qquad \text{Where } f'(x) = 0 \text{ or dne: } x = -1,\ x = 0,\ x = 1$$

$$f'(-2) = \frac{-}{+} < 0,\ f'(-\frac{1}{2}) = \frac{-}{+} < 0,\ f'(\frac{1}{2}) = \frac{+}{+} > 0,\ f'(2) = \frac{+}{+} > 0$$



Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

6. **Potential inflection points:** Find critical numbers of f'
7. **Concavity**: On intervals to each side of points where $f''(x) = 0$ or d.n.e., find whether f is $\smile$ or $\frown$. If the concavity changes at a point where f exists, f has an infl pt there: if f' d.n.e, it's an infl pt w/ a vert tangent, if $f'(x) = 0$, then it's an infl pt w/ a hor tangent.

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

6. **Potential inflection pts:** Find where $f''(x) = 0$ or dne (but f' does).

$f'(x) = \frac{12x}{(x^2-1)^2}$

$$
\begin{aligned}
f''(x) &= \frac{(x^2-1)^2 \cdot 12 - 12x(2)(x^2-1)(2x)}{(x^2-1)^2} \\
&= \frac{(x^2-1)\Big(12(x^2-1) - 48x^2\Big)}{(x^2-1)^4} \\
&= \frac{12x^2 - 12 - 48x^2}{(x^2-1)^3} = \frac{-12(3x^2+1)}{(x^2-1)^3}
\end{aligned}
$$

$f''(x)$ dne at $x = \pm 1$ (same as f & f').

$f''(x) = 0 \iff$ numerator$= 0$, which it doesn't, so f has no infl pts.

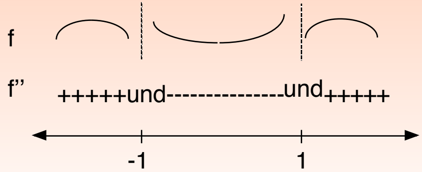
However, concavity *could* change at the asymptotes.

Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

7. **Concavity**: On intervals to each side of points where $f''(x) = 0$ or d.n.e., find whether f is $\smile$ or $\frown$. If the concavity changes at a point where f exists, f has an infl pt there: if f' d.n.e, it's an infl pt w/ a vert tangent, if $f'(x) = 0$, then it's an infl pt w/ a hor tangent.

$f''(x) = \frac{-12(3x^2+1)}{(x^2-1)^3}$

$$f''(-2) = \frac{-}{+} < 0 f''(0) = \frac{-}{-} > 0 f''(2) = \frac{-}{+} < 0$$



Graphing $f(x) = \dfrac{2x^2 - 8}{x^2 - 1}$

8. **Horizontal Asymptotes**: Check the limit of $f(x)$ as $x \to \pm\infty$. If either limit is finite, you have a horizontal asymptote on that side. Polynomials and roots do not have horizontal asymptotes.
   A function is most likely to have a horizontal asymptote if it involves an exponential function or is a quotient of two functions.

9. **A few key points:** Find the y-values of all the points you've identified as important. Find the y-intercept, and if it's not too painful, find the x-intercepts as well.

In Class Work

1. Sketch the graph of $f(x) = x\ln(x)$, labeling and discussing all significant features.

Solutions:

1. Sketch the graph of $f(x) = x\ln(x)$, labeling and discussing all significant features.

- **Domain:** $f(x)$ is undefined for all $x \leq 0$
- **Vertical asymptotes:** $\ln(x)$ has a vertical asymptote at $x = 0$. Does this function?

$$\lim_{x\to 0^+} x\ln(x) \overset{0\cdot -\infty}{=} \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \overset{-\infty/\infty}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

Thus f does **not** have a vertical asymptote, but instead approaches the point $(0, 0)$.

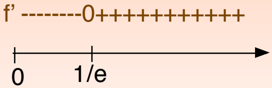
Solutions:

- Domain is all $x > 0$; $\lim\limits_{x \to 0^+} f = 0$
- **Critical Numbers:**

$$f'(x) = x\left(\frac{1}{x}\right) + \ln(x)(1) = 1 + \ln(x)$$

  $f'(x)$ is undefined for all $x \leq 0$, as is f.
  $f'(x) = 0$ when $\ln(x) = -1$, or when $x = e^{-1} = \frac{1}{e}$.
  Thus the only critical number is $x = \frac{1}{e}$.
- **Increasing/Decreasing:**

$$f'(1/4) = \ln(1/4) + 1 < 0 \qquad f'(1) = \ln(1) + 1 > 0$$



- **Local Extrema:**
  f has a local minimum at $x = \frac{1}{e}$

Solutions:

- Domain is all $x > 0$; $\lim\limits_{x\to 0^+} f = 0$
- $f \downarrow$ on $\left(0, \frac{1}{e}\right)$, $\uparrow$ on $\left(\frac{1}{e}, \infty\right)$; f has a local min at $x = \frac{1}{e}$
- **Potential Inflection Points:**

  $$f''(x) = \frac{1}{x}.$$

  $f''(x)$ d.n.e. at $x = 0$, (but $x = 0$ isn't in domain)
  $f''(x) \neq 0$.
  Thus f has no inflection points.
- **Concavity:**
  No inflection points $\Rightarrow$ f is $\smile$ or $\frown$ everywhere.
  Find sign of f'' at one point in domain to know concavity everywhere.

  $$f''(1) = \frac{1}{1} > 0 \Rightarrow f \text{ concave up on entire domain}$$

Solutions:

- Domain is all $x > 0$; $\lim_{x \to 0^+} f = 0$
- $f \downarrow$ on $\left(0, \frac{1}{e}\right)$, $\uparrow$ on $\left(\frac{1}{e}, \infty\right)$; f has a local min at $x = \frac{1}{e}$
- f is concave up on $(0, \infty)$
- **Horizontal asymptotes?**

$$\lim_{x \to \infty} x \ln(x) = \infty \cdot \infty = \infty.$$

Thus f has no horizontal asymptotes

Solutions:

- Domain is all $x > 0$; $\lim_{x \to 0^+} f = 0$
- $f \downarrow$ on $\left(0, \frac{1}{e}\right)$, $\uparrow$ on $\left(\frac{1}{e}, \infty\right)$; f has a local min at $x = \frac{1}{e}$
- f is concave up on $(0, \infty)$
- $\lim_{x \to \infty} f = \infty$
- **A few key points:**
  Only significant points we've found so far:
  - our local minimum at $x = 1/e$:

$$f(1/e) = \frac{1}{e} \ln\left(\frac{1}{e}\right) = \frac{1}{e} \ln(e^{-1}) = -\frac{1}{e}$$

  - y-intercept? $x = 0$ is not in the domain. But we *do* know that f approaches $y = 0$ as $x \to 0^+$.
  - x-intercept? Where is $x \ln(x) = 0$? Since $x \neq 0$, only possibility is where $\ln(x) = 0$, which is at $x = 1$.

Solutions:

- Domain is all $x > 0$; $\lim_{x \to 0^+} f = 0$
- $f \downarrow$ on $\left(0, \frac{1}{e}\right)$, $\uparrow$ on $\left(\frac{1}{e}, \infty\right)$; f has a local min at $x = \frac{1}{e}$
- f is concave up on $(0, \infty)$
- $\lim_{x \to \infty} f = \infty$
- $f(1/e) = -1/e$, $f(1) = 0$