Find the following integrals, and *check your answers!!*

1. $\displaystyle\int_2^5 \frac{1}{x\ln(x)}\,dx$ Hint: Let $u = \ln(x)$

   Let $u = \ln(x)$. Then $\dfrac{du}{dx} = \dfrac{1}{x}$, so $du = \dfrac{1}{x}\,dx$. Looking at the original integral, I see I can rewrite it as

   $$\int_2^5 \frac{1}{x\ln(x)}\,dx = \int_2^5 \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx.$$

   Thus I can substitute du in directly for $\dfrac{1}{x}\,dx$, and u in for $\ln(x)$, and I get

   $$\int_2^5 \frac{1}{x\ln(x)}\,dx = \int_{x=2}^{x=5} \frac{1}{u}\,du.$$

   Once again, this eliminated all the x terms from the integral and produced something simpler that I know how to antidifferentiate:

   $$\begin{aligned}\int_{x=2}^{x=5} \frac{1}{u}\,du &= \ln(u) \text{ from } x = 2 \text{ to } x = 5 \\ &= \ln(\ln(x)) \text{ from 2 to 5} \\ &= \ln(\ln(5)) - \ln(\ln(2))\end{aligned}$$

   Check: Differentiate the antiderivative:

   $$\frac{d}{dx}\left(\ln(\ln(x))\right) = \frac{1}{\ln(x)} \cdot \frac{1}{2} = \frac{1}{x\ln(x)},$$

   which is the integrand I started with, so I know I at least found the correct antiderivative.

2. $\displaystyle\int \sin(x)e^{\cos(x)}\,dx$

   I look at this, and I see a product, with one of the terms being a composition. I know that this comes from the chain rule.

   Since substitution undoes the chain rule, we generally choose u to be the inside function in the composition.

Here, the composition is $e^{\cos(x)}$, and the inside function is $\cos(x)$.

Choose $u = \cos(x)$. Then $\dfrac{du}{dx} = -\sin(x)$, so $du = -\sin(x)\ dx$.

When I look back at the integrand, $\sin(x)e^{\cos(x)}$, I see that while there is a $\sin(x)$ in the integrand, there is no minus sign. Therefore, I'll multiply both sides by -1:

$$-du = \sin(x)\ dx.$$

Now I'm ready to replace $\cos(x)$ by u, and $\sin(x)\ dx$ by $-du$:

$$\begin{aligned}
\int \sin(x)e^{\cos(x)}\ dx &= \int e^u \cdot -1\ du \\
&= -\int e^u\ du \\
&= -e^u + C \\
&= -e^{\cos(x)} + C
\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.

3. $\displaystyle\int x^2 \sec(x^3)\tan(x^3)\ dx$

Once again, I see a product, where one of the terms (if I think of $\sec(x^3)\tan(x^3)$ as just one term!) is a composition.

Therefore, once again, I'm thinking I need to undo the chain rule, so I'm thinking substitution.

And again, I let u be the inside function,which would be x^3.

Let $u = x^3$. Then $\dfrac{du}{dx} = 3x^2$, so $du = 3x^2\ dx$. Comparing this with the integrand $x^2\sec(x^3)\tan(x^3)$, I see that the integrand doesn't have $3x^2$, but it does have x^2. (It's important that the terms involving x's are present – we can adjust for multiplicative constants!)

I can adjust my du term by multiplying both sides by $\dfrac{1}{3}$:

$$\frac{1}{3}\ du = x^2\ dx.$$

Replacing x^3 with u and $x^2\,dx$ with $\frac{1}{3}\,du$, I see that I now have

$$\begin{aligned}
\int x^2 \sec(x^3)\tan(x^3)\,dx &= \int \sec(u)\tan(u)\cdot\frac{1}{3}\,du \\
&= \frac{1}{3}\int \sec(u)\tan(u)\,du \\
&= \frac{1}{3}\sec(u) + C \\
&= \frac{1}{3}\sec(x^3) + C
\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.

4. $\int e^x \cos(e^x)\,dx$

Again, I see a product where one of the terms is a composition, so again, I'm thinking substitution.

Let u be the inside function, so let $u = e^x$. Then $\frac{du}{dx} = e^x$, so $du = e^x\,dx$.

When I look back at the integrand, I see that there are two copies of e^x present, which is a good thing! The one inside the cosine function will be u, while the one outside is necessary to be part of du!

$$\begin{aligned}
\int e^x \cos(e^x)\,dx &= \int \cos(u)\,du \\
&= \sin(u) + C \\
&= \sin(e^x) + C
\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.

5. $\int e^x e^{2x}\,dx$ Hint: $e^{2x} = (e^x)^2$

There are at least two ways to do this problem!

- **Method 1:**

$$\int e^x e^{2x}\ dx = \int e^{3x}\ dx.$$

Let $u = 3x$. Then $\dfrac{du}{dx} = 3$, so $du = 3dx$, so $\dfrac{1}{3}du = dx$.
Thus

$$\begin{aligned}
\int e^x e^{2x}\ dx &= \int e^{3x}\ dx \\
&= \int e^u \cdot \frac{1}{3}\ du \\
&= \frac{1}{3}\int e^u\ du \\
&= \frac{1}{3}e^u + C \\
&= \frac{1}{3}e^{3x} + C
\end{aligned}$$

- **Method 2:**

If we look at the integral the way it was originally given to us,

$$\int e^x e^{2x}\ dx = \int e^x (e^x)^2\ dx,$$

we might think of using $u = e^x$, in which case, $du = e^x\ dx$, so we'd have

$$\begin{aligned}
\int e^x e^{2x}\ dx &= \int e^x (e^x)^2\ dx \\
&= \int u^2\ du \\
&= \frac{1}{3}u^3 + C \\
&= \frac{1}{3}(e^x)^3 + C \\
&= \frac{1}{3}e^{3x} + C
\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.

6. $\displaystyle\int \frac{(\sec(x))^2}{\tan(x)}\,dx$

This one is a bit tricky. The obvious composition is $(\sec(x))^2$. And yet division is also a composition. So there's definitely composition going on, but what u should be isn't immediately obvious ... at least not when thinking about what the inside function is.

So instead, look to see if any piece looks like the derivative of any *other* piece. When we look at it that way, we see immediately that we have a $\tan(x)$, and we have $\sec^2(x) = \dfrac{d}{dx}(\tan(x))$.

That suggests that we might try letting $u = \tan(x)$. Of course, we don't know if it's going to work yet, but it's a place to start.

Let $u = \tan(x)$. Then $\dfrac{du}{dx} = \sec^2(x) = (\sec(x))^2$, so $du = (\sec(x))^2\,dx$. Substituting in, we get

$$\begin{aligned}\int \frac{(\sec(x))^2}{\tan(x)}\,dx &= \int \frac{1}{u}\,du\\ &= \ln(u) + C\\ &= \ln(\tan(x)) + C\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.

7. $\displaystyle\int x\sqrt{x-1}\,dx$ Hint: Let $u = x - 1$

Despite it's deceptively simply appearance, in some ways this is a challenging one. It's not immediately obvious how this choice of u is going to help us:

Let $u = x - 1$. Then $du = dx$ ... and what do we do with that extra x sitting out front? We can't mix u's and x's, after all ...

Still, it's not hard to solve for x in terms of u: if $u = x - 1$, then $x = u+1$. Even after having that insight, though, we don't immediately feel as if we've accomplished much: In

$$\int x\sqrt{x-1}\,dx = \int (u+1)\sqrt(u)\,du,$$

all we've really done is switched around where the addition/substraction is going on. It turns out, though, that that's all we needed to do! we can distribute the $\sqrt{u}$ term throughout the $u+1$ part:

$$\begin{aligned}
\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt(u)\,du \\
&= \int u\sqrt{u} + \sqrt{u}\,du \\
&= \int u^{3/2} + u^{1/2}\,du \\
&= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C
\end{aligned}$$

Check by differentiating this antiderivative to see make sure we get the original integrand.