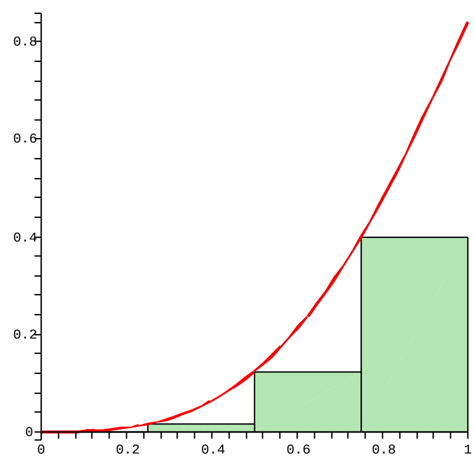
Let $I = \int_0^1 x \sin(x^2)\ dx$

1. Calculate L_4 by hand.
   Does this overestimate or underestimate I?

   - Subdivide $[0, 1]$ in to 4 equal subintervals:
     $$\Delta x = \frac{1-0}{4} = \frac{1}{4},$$
     so the subintervals will be
     $$[0, \frac{1}{4}] \qquad [\frac{1}{4}, \frac{1}{2}] \qquad [\frac{1}{2}, \frac{3}{4}] \qquad [\frac{3}{4}, \frac{1}{}].$$
   - Sketch a picture: Because this is a left sum, the heights of the four rectangles will be determined by the heights of the function at the left-hand endpoints of each subintervals. That is, we'll be approximating the integral with the rectangles shown below:



   - Write the sum:

$$\begin{aligned} L_4 &= \text{sum of the areas of the four rectangles} \\ &= b_1 h_1 + b_2 h_2 + b_3 h_3 + b_4 h_4 \end{aligned}$$

Each of the four rectangles has base= $\frac{1}{4}$.

- Left-most rectangle has height $h_1 = f(\text{left-hand endpoint of } [0, \frac{1}{4}]) = f(0)$.
- Next rectangle has height $h_2 = f(\text{left-hand endpoint of}[\frac{1}{4}, \frac{1}{2}]) = f(\frac{1}{4})$
- Next rectangle has height $h_3 = f(\frac{1}{2})$
- Right-most rectangle has height $h_4 = f(\frac{3}{4})$

$$
\begin{aligned}
L_4 &= \frac{1}{4}(f(0) + f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4}) \\
&= \frac{1}{4}(0 + \frac{1}{4}\sin(\frac{1}{16}) + \frac{1}{2}\sin(\frac{1}{4}) + \frac{3}{4}\sin(\frac{9}{16})) \\
&= .1348234536
\end{aligned}
$$

Because f is increasing over this interval, the function will always be lower at the left-hand side of a subinterval than on the right-hand side. That means that the height of the rectangles is determined by the lowest point on the subinterval, which means (as we can easily see in the above picture) that the left sum *under*-estimates the actual value of the integral.

2. Write L_4 using sigma notation.

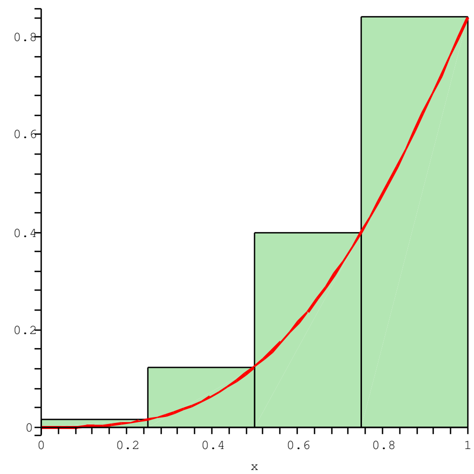
   In order to write this in sigma notation, I need to look for what varies from term-to-term and what stays the same.

   - Just as when I was writing L_4 out term-by-term, I can factor the base, $\frac{1}{4}$, out of the sum. (Remember, sigma notation is just short-hand for a very long sum).
   - It's usually easiest to begin by writing the heights as $f(?)$, and to replace f by what f actually is after we've done the hard part.
   - What varies is what's inside f.

- How does it vary? I begin by plugging in 0, then $\Delta x = \frac{1}{4}$, then $2\Delta x = \frac{1}{2}$, and finally $3\Delta x = \frac{3}{4}$.
  Because I begin by plugging in 0, I think it's going to be easiest to see a pattern if I begin numbering the terms of the sum with the "zero-th" term. (This is not the *only* way, it's just the *easiest* way.)
  If I do that, then in the 0th term, I plug in $0 \cdot \Delta x$, in the 1st term, I plug in $1 \cdot \Delta x$, in the 2nd term, I plug in $2 \cdot \Delta x$, and in the 3rd (and last) term, I plug in $3 \cdot \Delta x$.
  Thus the number of the term (with this numbering) corresponds to what multiple of Δx I'm plugging in.
-
$$\begin{aligned} L_4 &= \frac{1}{4}\sum_{i=0}^{3} f(i\Delta x) \\ &= = \frac{1}{4}\sum_{i=0}^{3} f(\frac{i}{4}) \\ &= \frac{1}{4}\sum_{i=0}^{3} \frac{i}{4}\sin((\frac{i}{4})^2) \end{aligned}$$

3. Calculate R_4 by hand.
   Does this overestimate or underestimate I?

- Subdivide $[0, 1]$ in to 4 equal subintervals:
  These will of course be the same subintervals as for L_4:
$$[0, \frac{1}{4}] \qquad [\frac{1}{4}, \frac{1}{2}] \qquad [\frac{1}{2}, \frac{3}{4}] \qquad [\frac{3}{4}, \frac{1}{]}.$$
- Sketch a picture:
  Because this is a right sum, the heights of the four rectangles will be determined by the heights of the function at the right-hand endpoints of each subintervals. That is, we'll be approximating the integral with the rectangles shown below:



- Write the sum:

  Notice that all but the right-most rectangle showed up in the left sum, too – only in the left sum, they were to the right of where they are now.

$$\begin{aligned} R_4 &= \text{sum of the areas of the four rectangles} \\ &= b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4 \end{aligned}$$

  Each of the four rectangles has base$= \frac{1}{4}$, just as before.

  - Left-most rectangle has height $h_1 = f(\text{right-hand endpoint of } [0, \frac{1}{4}]) = f(\frac{1}{4})$.
  - Next rectangle has height $h_2 = f(\text{right-hand endpoint of}[\frac{1}{4}, \frac{1}{2}]) = f(\frac{1}{2})$
  - Next rectangle has height $h_3 = f(\frac{3}{4})$
  - Right-most rectangle has height $h_4 = f(1)$

$$\begin{aligned} R_4 &= \frac{1}{4}(f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4} + f(1)) \\ &= \frac{1}{4}(\frac{1}{4}\sin(\frac{1}{16}) + \frac{1}{2}\sin(\frac{1}{4}) + \frac{3}{4}\sin(\frac{9}{16}) + 1\sin(1)) \\ &= .3451911998 \end{aligned}$$

Because f is increasing over this interval, the right sum *over*-estimates the actual value of the integral.

4. Write R_4 using sigma notation.

   Again, I need to look for what varies from term-to-term and what stays the same.

   - Factor the base, $\frac{1}{4}$, out of the sum.
   - Begin by writing the heights in terms of f.
   - What varies is what's inside f.
   - How does it vary? I begin by plugging in $\Delta x = \frac{1}{4}$, then $2\Delta x = \frac{2}{4}$, then $3\Delta x = \frac{3}{4}$, and finally $4\Delta x = \frac{4}{4} = 1$.

     I notice that the terms I'm plugging in are multiples of $\frac{1}{4}$, and that in the first term (notice I decided to call it the "first" term rather than the "zero-th" term. That's because I'm plugging in $\frac{\mathbf{1}}{4}$ rather than 0.) ... anyway, in the first term, I'm plugging in $\frac{\mathbf{1}}{4}$, in the 2nd term I'm plugging in $\frac{\mathbf{2}}{4}$, in the 3rd term I'm plugging in $\frac{\mathbf{3}}{4}$ and in the fourth term I'm plugging in $\frac{\mathbf{4}}{4}$.
     Thus the number of the term (with this numbering) corresponds to what multiple of Δx I'm plugging in.

•

$$
\begin{aligned}
R_4 &= \frac{1}{4}\sum_{i=1}^{4} f(i\Delta x) \\
&= = \frac{1}{4}\sum_{i=1}^{4} f(\frac{i}{4}) \\
&= \frac{1}{4}\sum_{i=1}^{4} \frac{i}{4}\sin((\frac{i}{4})^2)
\end{aligned}
$$

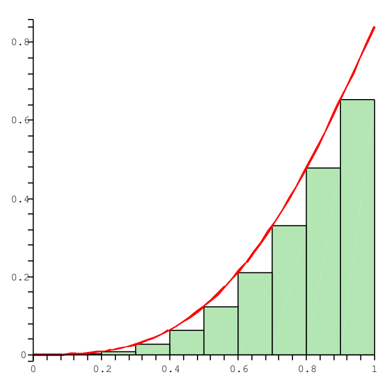
When you compare this to the result for L_4, you'll see everything's *exactly* the same, *except* for the indices of the sum. In both cases, we're dealing with four terms. In the left sum, we count those four terms $\{0, 1, 2, 3\}$, and in the right sum, we count them $\{1, 2, 3, 4\}$. Everything else is *exactly* the same – I didn't have to do *anything* to adjust for dealing with the right endpoint of each interval instead of the left except change the indices.

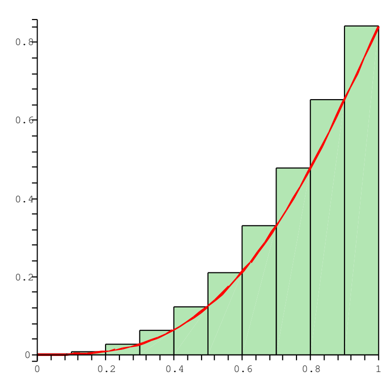
5. Use Maple to draw L_{10} and R_{10}.
(Use the leftbox() and rightbox() commands)

Type:

```
with(student):
leftbox(x*sin(x^2), x=0..1, 10);
rightbox(x*sin(x^2),x=0..1,10);
```

You will get these pictures:





6. Use Maple to calculate L_{10} and R_{10}.
   (Use the leftsum() and rightsum() commands)

   How does I compare to L_{10} and R_{10}?

   First of all, this isn't all that hard to do by hand, once we've done it for L_4 an R_4.

   What's going to change when we go from 4 terms to 10?

   - $\Delta x = \frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10}$
   - We'll be adding up 10 terms instead of 4.
   - The inputs into the functions will of course change, since those depend on the endpoints of the intervals, which depends on how many subintervals there are. Instead of the endpoints of the all the subintervals (the *partition*) being

     $$\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\},$$

     it will be

     $$\{0, \frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \ldots, \frac{9}{10}, 1\}.$$

   - Thus

   $$\begin{aligned} L_{10} &= \frac{1}{10}(f(0) + f(\frac{1}{10}) + f(\frac{2}{10}) + f(\frac{3}{10}) + \ldots + f(\frac{8}{10}) + f(\frac{9}{10})) \\ &= .1893808149 \\ R_{10} &= \frac{1}{10}(f(\frac{1}{10}) + f(\frac{2}{10}) + f(\frac{3}{10}) + \ldots + f(\frac{8}{10}) + f(\frac{9}{10}) + f(1)) \\ &= .2735279134 \end{aligned}$$

   We know that because $f(x) = x\sin(x^2)$ is increasing, L_{10} under-estimates the integral and R_{10} over-estimates the integral. Thus we know that

   $$.1893808149 \leq \int_0^1 x\sin(x^2)\, dx \leq .2735279134.$$

   But if I want to do this with Maple (if I've already typed in "with(student)"), I simply type in

```
leftsum(x*sin(x^2), x=0..1, 10);
evalf(%);
rightsum(x*sin(x^2), x=0..1, 10);
evalf(%);
```

7. Find the exact value of I by using u-substitution. Does this make sense?

   When I look at $\int_0^1 x\sin(x^2)\ dx$, I see that the integrand is a product, and that one of the terms is a composition. I am therefore thinking this is a prime candidate to practice substition on.

   As always, I let $u =$ the inside $= x^2$. Then differentiating u, we find that $\frac{du}{dx} = 2x$, so $du = 2xdx$. I therefore know that $\frac{1}{2}du = x\ dx$.

   Replacing the $\sin(x^2)$ term in the integral with $\sin(u)$ and the $x\ dx$ term in the integral with $\frac{1}{2}\ du$, I find that the original integral is equivalent to the following:

$$\begin{aligned}
\int_0^1 x\sin(x^2)\ dx &= \int_{x=0}^{x=1} \frac{1}{2}\sin(u)\ du \\
&= \left[-\frac{1}{2}\cos(u)\right] \text{ from } x=0 \text{ to } x=1 \\
&= \left[-\frac{1}{2}\cos(x^2)\right] \text{ from } x=0 \text{ to } x=1 \\
&= -\frac{1}{2}\cos(1) + \frac{1}{2}\cos(0) \\
&= .2298488470
\end{aligned}$$

   Sure enough, as predicted, this result is between L_{10} and R_{10}!