1. Let $I = \displaystyle\int_{1.5}^{2} \sin(x^2)\,dx$

   (a) Write L_5 out term-by-term and in sigma notation, without using Maple. (That is, only use Maple to calculate specific terms, don't just use the `leftsum` command.)
   $\Delta x = \dfrac{2-1.5}{5} = \dfrac{.5}{5} = .1.$

   $$L_5 = .1\sum_{k=0}^{4} f(1.5+.1k) = .1\sum_{k=0}^{4}\sin((1.5+.1k)^2) = .1026661074.$$

   (b) Write R_5 term-by-term and in sigma notation, without using Maple.

   $$R_5 = .1\sum_{k=1}^{5} f(1.5{+}.1k) = .1\sum_{k=1}^{5}\sin((1.5{+}.1,)^2) = -0.05082146181.$$

   (c) Using Maple, calculate L_{10}.
   Will this overestimate or underestimate I?
   Using Maple, I find that

   $$L_{10} = .05\sum_{k=0}^{9} f(1.5{+}.05k) = .05\sum_{k=0}^{9}\sin((1.5{+}.05k)^2) = 0.06475798670.$$

   Because the integrand is decreasing, and so the left-side of the graph is higher than the right, the rectangles will be above the graph, and so it over-estimates.

   (d) Using Maple, calculate R_{10}.
   Will this overestimate or underestimate I? Using Maple, I find that

   $$R_{10} = .05\sum_{k=1}^{10} f(1.5{+}.05k) = .05\sum_{k=1}^{10}\sin((1.5{+}.05k)^2) = -0.01198579788.$$

   Because the integrand is decreasing, and so the right-side of the graph is lower than the left, the rectangles will be below the graph, and so it under-estimates.

(e) How accurate are your approximations to the true value of I? Tell me as much as you can.

Well, I don't know the true value of I. Even if I ask Maple to integrate for me, I'm only going to get an approximation.

One thing I do know, though. Because the right sum is an under-estimate, and the left sum is an over-estimate, I know that

$$-0.01198579788 \leq \int_{1.5}^{2} \sin(x^2)\, dx \leq 0.06475798670.$$

That means that the value of the integral can be no more than

$$0.06475798670 - (-0.01198579788) = 0.07674378458$$

away from either of our approximations.

(f) Approximate I accurate within 0.001. You may use Maple to do this, but only use the ideas of left and right sums – don't use Maple to calculate the definite integral as that will only be an approximation anyway.

So if I can find a value of n so that $L_n - R_n \leq .001$, both L_n and R_n will be within .001 of $\int_{1.5}^{2} \sin(x^2)\, dx$.

So I just experiment with Maple. (After being sure to have typed "with(student);" at some point), I type in

```
leftsum(sin(x^2), x=1.5..2, 10)-rightsum(sin(x^2), x=1.5..2, 10);
evalf(%);
```

I already know the value of this – it's what I figured out just now. But by changing the two 10s to higher numbers, I should get the difference closer and closer to .001.

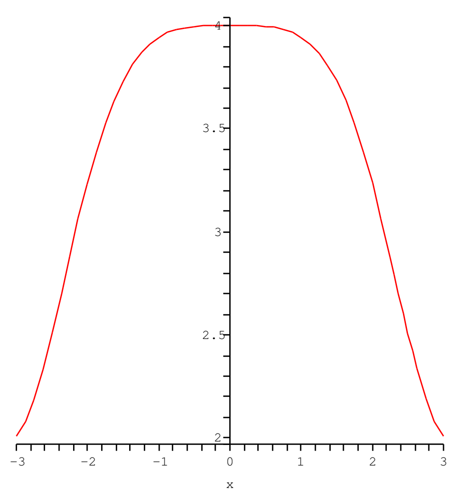
I experimented, and found that anything over 800 terms would work. But then, because I'm obsessive, I decided to see exactly at what point the two sums went from being further than .001 apart to being less than .001 apart. I determined that in fact, if you want to be picky, if you use anything more than 767 terms, you are guaranteed that both the left and right sums are within .001 of the actual value of the integral.

Now remember, we got that by figuring out where the left and right sums are .001 apart. Since the actual value of the integral is somewhere in between, in truth we probably get to within .001 by adding considerably fewer terms. However, since we can't *find* the value of the integral exactly, we can't know how many terms this is. So we just take what we can get.

2. Let $I = \int_{-3}^{3} \cos\left(\frac{x^2}{3}\right) + 3\,dx$.

   Use the same ideas developed above to pproximate I accurate within 0.02 of its actual value.

   When I graph the integrand, I see that it is neither uniformly increasing nor decreasing on this integral.



   It's increasing for the first half of the interval, then decreasing for the second half. But the idea of squeezing the integral in between the left and right sums only works if we *know* one is bigger and the other smaller, which we only know if our function is either always increasing or always decreasing.

   It occurs to me to divide the interval in half.

$$\int_{-3}^{3} \cos\left(\frac{x^2}{3}\right) + 3\,dx = 2\int_{0}^{3} \cos\left(\frac{x^2}{3}\right) + 3\,dx.$$

On the interval $[0, 3]$, my function is only decreasing, so the ideas I used above will work.

So I again work with the difference between the left and right sums, increasing the number of terms I need to use until I get that the difference is less than .02.

What I found was that when I used:

```
g:= x -> x-> cos(x^2/3)+3;
leftsum(g(x), x=0..3, 299)-rightsum(g(x), x=0..3, 299);
evalf(%);
```

I got that the difference was less than .02, but if I used any smaller number of terms, the difference was greater.

Thus, while more than likely I could use slightly fewer terms and be within .02 of the actual value of the integral, the only way I can guarantee that I'm within .02 of the actual value (using this technique) is to use either L_{299} or R_{299}.