1. Find the derivatives of the following functions, and verify your answer by graphing f and f' on the same set of axes.

   (a) $f(x) = x^2 - 2x + 3$

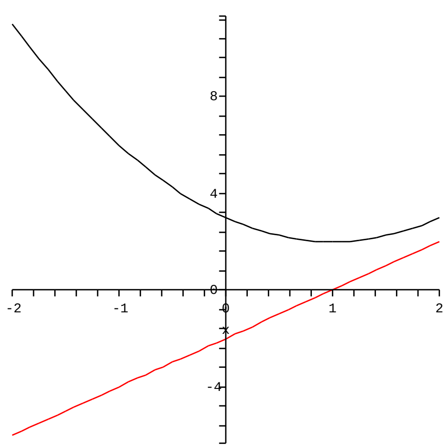
   Combining the following facts:

   - $\frac{d}{dx}(x^n) = nx^{n-1}$ for all n
   - $\frac{d}{dx}(kf(x)) = kf'(x)$
   - $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
   - $\frac{d}{dx}(k) = 0$

   I find that

$$\begin{aligned} f'(x) &= 2x^1 - 2(1x^0) + 0 \\ &= 2x - 2 \end{aligned}$$

   Let's verify this result:



   We can see that where $f(x)$ (in black) switches from decreasing to increasing at $x = 1$, $f'(x)$ (in red) switches from being negative to positive, just as we'd expect. We can also see that $f(x)$ is always concave up on the interval we're looking at (in fact, it always is), and just as expected, $f'(x)$ is always increasing.

   Thus we have the graphical relationship we'd expect, verifying that we more than likely found the correct derivative.

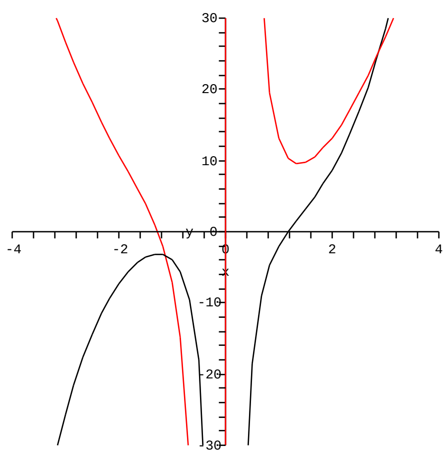
(b) $f(x) = x^3 - \dfrac{5}{x^2} + 2$

The only type of functions we currently know how to differentiate are those of the form x^n, and linear combinations of such functions. Thus in order to differentiate the portion of the above function $\dfrac{5}{x^2}$, we must first make sure it is in the form kx^n.

Remembering that $\dfrac{1}{x^n} = x^{-n}$ is the key to doing such a function. Thus,

$$\begin{aligned} f(x) &= x^3 - 5x^{-2} + 2 \\ \Rightarrow f'(x) &= 3x^2 - 5(-2x^{-3}) + 0 \\ &= 3x^2 + 10x^{-3} \end{aligned}$$

Again, let's verify by graphing:

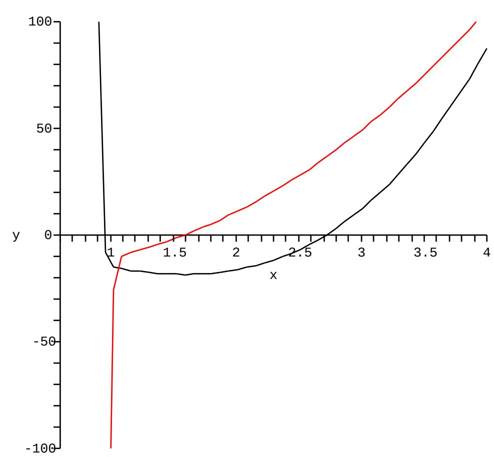


f (in black again) increases on roughly $[-4, -1.2]$ and $(0, 4)$, and we see that f' (in red) is positive on those same intervals, just as it should be. f is concave down on $(-4, 0)$ and roughly $(0, 1)$, and f' is decreasing over those same intervals, just as it should be. We again have the expected graphical result.

(c) $f(x) = 2x^{\pi} + x^{-42} - 17x$
The only aspect of this function that's different from the previous two is the presence of π in the power. The whole key here, of course, is remembering that π is just another constant, so we differentiate x^{π} just the same as we would any other x^n.

$$\begin{aligned} f'(x) &= 2(\pi x^{\pi-1}) + (-42)x^{-43} - 17(1x^0) \\ &= 2\pi x^{\pi-1} - 42x^{-43} - 17 \end{aligned}$$

Again, verify this graphically:



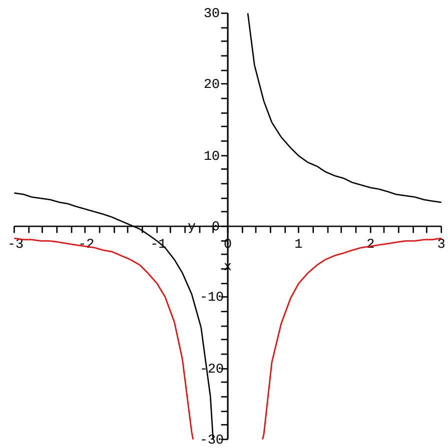
$f(x)$ is decreasing up to about $x = 1.5$, and $f'(x)$ (in red) is negative on that same interval. $f(x)$ appears to be always concave up (it's hard to tell just below 1, as the function is decreasing so steeply), and $f'(x)$ is always increasing. Again, our graphs have helped us check our results.

(d) $f(x) = \dfrac{7}{x} - x + 4$

As in Part (b), I rewrite $\dfrac{7}{x}$ as $7x^{-1}$.

$$\begin{aligned} f(x) &= 7x^{-1} - x + 4 \\ \Rightarrow f'(x) &= 7(-1x^{-2}) - 1x^0 + 0 \\ &= -7x^{-2} - 1 \end{aligned}$$

Once again, looking at the graphs of both f and f',



f is decreasing on the whole interval, and f' is, as expected, negative everywhere we can see. f is concave down on $[-3, 0)$, and f' is, again as expected, decreasing on $[-3, 0)$. It all fits together.

2. Find an *anti*derivative for each function in 1.

   (a) $f(x) = x^2 - 2x + 3$.

   We ask ourselves: what do we differentiate to get $f(x)$?

   Remember, since $\frac{d}{dx}(x^n) = nx^{n-1}$, an *antiderivative* of x^n is $\frac{x^{n+1}}{n+1}$.

   We can check that, by the way. We need to make sure that the derivative of $\frac{x^{n+1}}{n+1}$ is what we started with, x^n.

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\frac{d}{dx}(x^{n+1}) = \frac{1}{n+1}((n+1)x^{n+1-1}) = x^n.$$

   Since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, we know an antiderivative of x^n is indeed $\frac{x^{n+1}}{n+1}$.

   So, an antiderivative of $f(x)$ is

$$F(x) = \frac{x^{2+1}}{2+1} - 2\left(\frac{x^{1+1}}{1+1}\right) + 3\left(\frac{x^{0+1}}{0+1}\right) = \frac{1}{3}x^3 - x^2 + 3x.$$

   I can check this two ways: either graphically or by differentiating $F(x)$ to see if I get $f(x)$.

   Check:

$$F'(x) = \frac{1}{3}(3x^2) - 2x + 3 = x^2 - 2x + 3 = f(x).$$

(b) $f(x) = x^3 - \dfrac{5}{x^2} + 2$

$$\begin{aligned} f(x) &= x^3 - 5x^{-2} + 2x^0 \\ \Rightarrow F(x) &= \frac{x^4}{4} - 5\left(\frac{x^{-1}}{-1}\right) + 2\left(\frac{x^1}{1}\right) \\ &= \frac{1}{4}x^4 + \frac{5}{x} + 2x \end{aligned}$$

Check:

$$F'(x) = \frac{1}{4}(4x^3) + 5(-1x^{-2}) + 2 = x^3 - \frac{5}{x^2} + 2 = f(x).$$

(c) $f(x) = x^\pi + x^{-42} - 17x$

$$F(x) = \frac{x^{\pi+1}}{\pi+1} + \frac{x^{-42+1}}{-42+1} - 17\left(\frac{x^2}{2}\right) = \frac{1}{\pi+1}x^{\pi+1} - \frac{1}{41x^{41}} - \frac{17}{2}x^2.$$

Check:

$$F'(x) = \frac{1}{\pi+1}\left((\pi+1)x^\pi\right) - \frac{1}{41}(-41x^{-42}) - \frac{17}{2}(2x) = x^\pi + x^{-42} - 17 = f(x).$$

(d) $f(x) = 7x^{-1} - x + 4$

$$F(x) = 7\left(\frac{x^{-1+1}}{-1+1}\right) - \frac{x^2}{2} + 4x = \mathbf{?} - \frac{x^2}{2} + 4x.$$

Uh oh! Apparently, we can't antidifferentiate x^{-1} using the formula $\dfrac{x^{n+1}}{n+1}$!. That doesn't necessarily mean *no* function has x^{-1} as its derivative, but it certainly means no *power* function has x^{-1} as its derivative.

So for now, we're left wondering ... what's the deal with $1/x$???