1. Find the maximum and minimum values of

$$f(x) = -x^3 + 5x^2$$

on the interval $-2 \le x \le 4$.

The max and min values must occur either at critical points (stationary points and points where the derivative does not exist but the function does) or at the endpoints of the interval.

To find the critical points, I must take the derivative:

$$f'(x) = -3x^2 + 10x.$$

The derivative exists everywhere. It is zero where

$$-3x^2 + 10x = 0 \Longrightarrow x(-3x + 10) = 0 \Longrightarrow x = 0 \text{ or } x = 10/3.$$

Both 0 and 10/3 are in the interval $[-2, 4]$, so I just need to test the values of f at these four values:

x	$f(x)$
-2	-8+20=12
0	0
10/3	$-1000/27 + 500/9 = \dfrac{-1000 + 1500}{27} = \dfrac{500}{27} \approx 18.5$
4	-64+80=16

Thus $f(x)$ attains its maximum value of roughly 18.5 at $x = 10/3$ and its minimum value of 0 at $x = 0$.

2. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

Since this is going to be a rectangle, and both of its upper two vertices are on the parabola, it must go from $-x$ to x for some value of x. The question is, what value of x makes the area be a maximum?

What will the area be?

$$\begin{aligned} A &= \text{base}height \\ &= 2x \cdot f(x) \\ &= 2x(12 - x^2) \\ &= 24x - 2x^3 \end{aligned}$$

Do I know anything else? I know that $x > 0$, because I'm using that it's positive, in the base.

I also know that this is a downward-opening parabola, that intersects the x-axis at $x = \pm\sqrt{12}$. Since the *upper* two vertices are on the parabola, I only need to pay attention to the part of the parabola that's above the x-axis.

Thus I know x is on the interval $[0, \sqrt{12}]$, and I also know that when $x = 0$ or $x = \sqrt{12}$, the area will be 0. That is, the minimum value occurs at the endpoints.

So, here I go. The maximum value of the area must occur at a critical point (which would have to be a stationary point):

$$A'(x) = 24 - 6x^2 \Longrightarrow 24 = 6x^2 \Longrightarrow 4 = x^2 \Longrightarrow x = \pm 2.$$

Since x can't be -2, that leaves me with $x = 2$, and a maximum area of $48 - 16 = 32$.

3. Find the point(s) on the parabola $y = x^2 - 3$ that is closest to the origin.

   (Hint: Rather than mimimizing the distance to the origin, you can minimize the *square* of the distance. This will make the algebra easier.)

   A point on the parabola has coordinates $(x, x^2 - 3)$. The distance between two points is given by $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Therefore,

$$\begin{aligned}
d &= \sqrt{(x-0)^2+(y-0)^2} \\
&= \sqrt{x^2+(x^2-3)^2} \\
D &= d^2 \\
&= x^2+(x^2-3)^2 \\
D' &= 2x+2(x^2-3)(2x) \\
0 &= 2x+4x(x^2-3) \\
&= 2x+4x^3-12x \\
&= -10x+4x^3 \\
&= 2x(-5+2x^2))
\end{aligned}$$

Thus the stationary points are $x=0$ or $2x^2-5=0$, which happens at $x=\pm\sqrt{5/2}$.

I need to determine which of these gives the minimum. For that, I can either do the first or second derivative tests, depending on how hard taking the second derivative is.

It's not hard at all, so I'll do that:

$$\begin{aligned}
D'' &= -10+12x^2 \\
D''(0) &= -10 \\
D''(\pm\sqrt{5/2}) &= 20
\end{aligned}$$

Thus there's a local max at $x=0$ (which is by no means the global max) and local mins (which are the global mins) at $x=\pm\sqrt{5/2}$.