1. Let $f(x) = \ln(x) - \frac{x^2}{20}$

   (a) Write the equation of the line tangent to the graph of $y = f(x)$ at $x = 5$.

   To find the equation of any line, we need

   - Slope
   - Point

   - **Slope:** The slope of the line tangent to the graph of $y = f(x)$ at $x = 5$ is $f'(5)$.

   $$\begin{aligned} f'(x) &= \frac{1}{x} - \frac{x}{10} \\ f'(5) &= \frac{1}{5} - \frac{5}{10} \\ &= \frac{2}{10} - \frac{5}{10} \\ &= -\frac{3}{10} \end{aligned}$$

     Therefore $m = -3/10$.

   - **Point:** The only point we have a hope of knowing on the line tangent to the graph of $y = f(x)$ at $x = 5$ is $(5, f(5)) = (5, \ln(5) - \dfrac{25}{20}) = (5, \ln(5) - \dfrac{5}{4})$.

   Therefore the equation of the line tangent to $y = f(x)$ at $x = 5$ has equation

   $$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - (\ln(5) - \frac{5}{4}) &= -\frac{3}{10}(x - 5) \\ y &= -\frac{3}{10}x + \frac{3}{2} + \ln(5) - \frac{5}{4} \\ &= -\frac{3x}{10} + \ln(5) + \frac{1}{4} \end{aligned}$$

   (b) Find the maximum and minimum values of $f(x)$ on the interval $[1, 12]$.

As we've seen before, *because the function is differentiable on* $[1, 12]$, the maximum and minimum values can only occur at stationary points or at $x = 1$ or $x = 12$.

Stationary Points:

$$\begin{aligned} f'(x) &= \frac{1}{x} - \frac{x}{10} \\ f'(x) = 0 &\Rightarrow \frac{1}{x} = \frac{x}{10} \\ &\Rightarrow x^2 = 10 \\ &\Rightarrow x = \pm\sqrt{10} \end{aligned}$$

Thus the only stationary point in the interval $[1, 12]$ is $\sqrt{10}$, and so the only values of f I need to check are those at $x = 1$, $x = \sqrt{10}$ and $x = 12$.

x	$f(x)\,\ln(x) - \frac{x^2}{20}$
1	$\ln(1) - \frac{1}{20} = -\frac{1}{20} = -.05$
$\sqrt{10}$	$\ln(\sqrt{10}) - \frac{10}{20} = \ln(\sqrt{10}) - \frac{1}{2} = .65$
12	$\ln(12) - \frac{144}{20} = \ln(12) - \frac{36}{5} = -4.72$

So the maximum value is .65, and it occurs at $x = \sqrt{10}$. The minimum value is -4.72, and it occurs at $x = 12$.

2. Suppose that A is a constant. Verify that $y(x) = Ae^x - x - 1$ is a solution to the differential equation $yy' = y^2 + xy$.

   I'll begin with the left hand side, and see if I can get to the right hand side.

$$\begin{aligned} yy' &= (Ae^x - x - 1) \cdot \frac{d}{dx}(Ae^x - x - 1) \\ &= (Ae^x - x - 1) \cdot (Ae^x - 1) \\ &= y \cdot (Ae^x - 1) \\ &= y \cdot (Ae^x - x - 1 + x) \\ &= y \cdot (y + x) \\ &= y^2 + xy \end{aligned}$$

3. Remember what we just did:

- when a quantity $y(t)$ grows at a rate proportional to the amount present, it's growth is described by the differential equation $y' = ky$, where k is the constant of proportionality.
- The general solution to this D.E. is $y(t) = Ce^{kt}$.

(a) A mold grows at a rate proportional to the amount present. Initially, its weight is 2g; after 2 days, it weighs 5 g. How much does it weight after 8 days?

Let $w(t) =$ the weight of the mold in grams after t days.
Because it grows at a rate proportional to the amount present, $w' = kw$, and so $w(t) = Ce^{kt}$.

All that remains is to find k and C. To do that, we'll use the two additional pieces of information we have.

- $w(0) = 2$. Therefore,
$$2 = w(0) = Ce^0 = C \Rightarrow C = 2 \Rightarrow w(t) = 2e^{kt}.$$
- $w(2) = 5$. Therefore,
$$5 = w(2) = 2e^{5k} \Rightarrow \frac{5}{2} = e^{5k} \Rightarrow \ln(5/2) = 5k \Rightarrow k = \frac{\ln(5/2)}{5}.$$
Therefore,
$$w(t) = 2e^{(\ln(5/2)/5)t}.$$

(b) A new radioactive element, Wheatonium, is discovered. It's half-life is 72 hours – that is, at the end of any 72 hour period, half the atoms that were radioactive at the beginning of the period are still radioactive, and half have decayed into non-radioactive atoms. How long does it take for

i. three-fourths of the atoms that were radioactive at the beginning of the period to have decayed?
Half of it decays after 72 hours, leaving half of what you started with. After another 72 hours, half of *that* decays, leaving you with one quarter of what you started with – so 3/4 would have decayed in 144 hours.

ii. 99% of the atoms that were radioactive at the beginning of the period to have decayed?

Because the rate its decaying is proportional to the amount present, we know that $A = kA'$, so $A(t) = Ce^{kt}$.
In order to find C and k, we use the information we were given.

- Suppose that we started with A_0. Then

$$A_0 = A(0) = Ce^0 = C \Rightarrow A(t) = A_0 e^{kt}.$$

- We know that after 72 hours, we have half of that left, so

$$\begin{aligned} A(72) &= \frac{1}{2}A_0 \\ \Rightarrow \frac{1}{2}A_0 = A(72) &= A_0 e^{72k} \\ \Rightarrow \frac{1}{2} &= e^{72k} \\ \Rightarrow \ln(1/2) &= 72k \\ \Rightarrow k &= \frac{\ln(1/2)}{72}. \end{aligned}$$

Thus $A(t) = A_0 e^{t(\ln(1/2)/72)}$.

We want to find when we have only 1% of what we started with, or in other words, for what t $A(t) = .01A_0$.
Find t so that:

$$\begin{aligned} .01A_0 = A(t) &= A_0 e^{t\ln(1/2)/72} \\ \Rightarrow .01 &= e^{t\ln(1/2)/72} \\ \Rightarrow \ln(.01) &= t\ln(1/2)/72 \\ \Rightarrow \frac{72\ln(.01)}{\ln(1/2)} &= t. \end{aligned}$$