Part I: 1. Find the derivatives of the following functions,

(a) $f(x) = x^2 - 2x + 3$

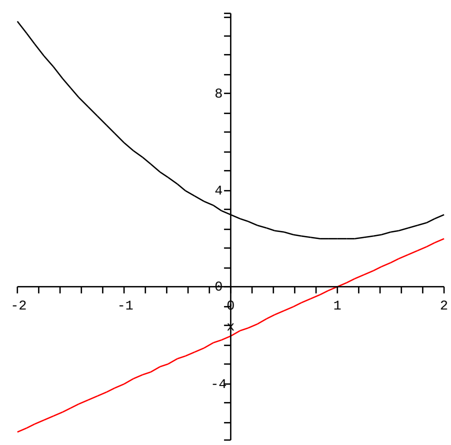
Combining the following facts:

- $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all n
- $\dfrac{d}{dx}(kf(x)) = kf'(x)$
- $\dfrac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
- $\dfrac{d}{dx}(k) = 0$

I find that

$$\begin{aligned} f'(x) &= 2x^1 - 2(1x^0) + 0 \\ &= 2x - 2 \end{aligned}$$

Let's verify this result:



We can see that where $f(x)$ (the top graph) switches from decreasing to increasing at $x = 1$, $f'(x)$ switches from being negative to positive, just as we'd expect. We can also see that $f(x)$ is always concave up on the interval we're looking at (in fact, it always is), and just as expected, $f'(x)$ is always increasing.

Thus we have the graphical relationship we'd expect, verifying that we more than likely found the correct derivative.

(b) $f(x) = x^3 - \dfrac{5}{x^2} + 2$

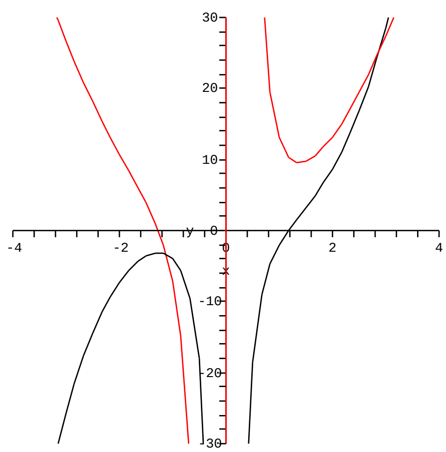
The only type of functions we currently know how to differentiate are those of the form x^n, and linear combinations of such functions. Thus in order to differentiate the portion of the above function $\dfrac{5}{x^2}$, we must first make sure it is in the form kx^n.

Remembering that $\dfrac{1}{x^n} = x^{-n}$ is the key to doing such a function.

Thus,

$$\begin{aligned} f(x) &= x^3 - 5x^{-2} + 2 \\ \Rightarrow f'(x) &= 3x^2 - 5(-2x^{-3}) + 0 \\ &= 3x^2 + 10x^{-3} \end{aligned}$$

Again, let's verify by graphing:

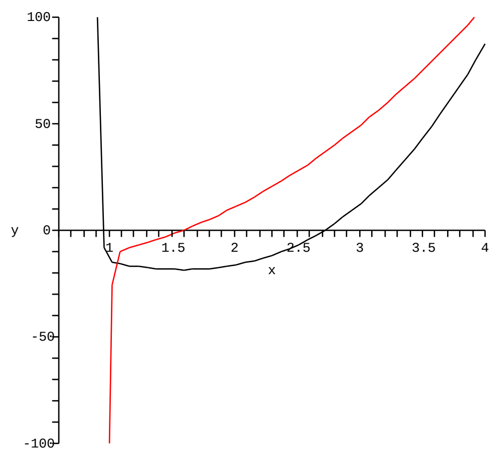


f increases on roughly $[-4, -1.2]$ and $(0, 4)$, and we see that f' is positive on those same intervals, just as it should be. f is concave down on $(-4, 0)$ and roughly $(0, 1)$, and f' is decreasing over those same intervals, just as it should be. We again have the expected graphical result.

(c) $f(x) = 2x^{\pi} + x^{-42} - 17x$
The only aspect of this function that's different from the previous two is the presence of π in the power. The whole key here, of course, is remembering that π is just another constant, so we differentiate x^{π} just the same as we would any other x^n.

$$\begin{aligned} f'(x) &= 2(\pi x^{\pi-1}) + (-42)x^{-43} - 17(1x^0) \\ &= 2\pi x^{\pi-1} - 42x^{-43} - 17 \end{aligned}$$

Again, verify this graphically:



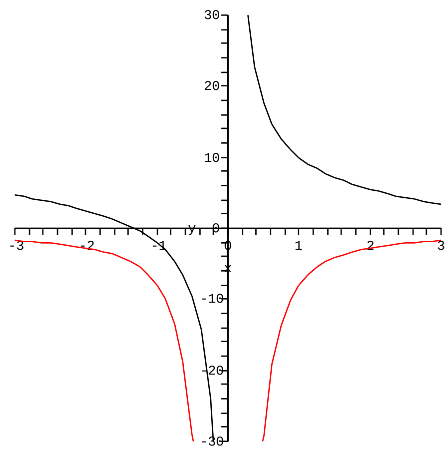
$f(x)$ is decreasing up to about $x = 1.5$, and $f'(x)$ is negative on that same interval. $f(x)$ appears to be always concave up (it's hard to tell just below 1, as the function is decreasing so steeply), and $f'(x)$ is always increasing. Again, our graphs have helped us check our results.

(d) $f(x) = \dfrac{7}{x} - x + 4$

As in Part (b), I rewrite $\dfrac{7}{x}$ as $7x^{-1}$.

$$\begin{aligned} f(x) &= 7x^{-1} - x + 4 \\ \Rightarrow f'(x) &= 7(-1x^{-2}) - 1x^0 + 0 \\ &= -7x^{-2} - 1 \end{aligned}$$

Once again, looking at the graphs of both f and f',



f is decreasing on the whole interval, and f' is, as expected, negative everywhere we can see. f is concave down on $[-3, 0)$, and f' is, again as expected, decreasing on $[-3, 0)$. It all fits together.

2. Find an *anti*derivative for each function in 1.

   (a) $f(x) = x^2 - 2x + 3$.
   We ask ourselves: what do we differentiate to get $f(x)$?
   Remember, since $\frac{d}{dx}(x^n) = nx^{n-1}$, an *antiderivative* of x^n is $\frac{x^{n+1}}{n+1}$.
   We can check that, by the way. We need to make sure that the derivative of $\frac{x^{n+1}}{n+1}$ is what we started with, x^n.

   $$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\frac{d}{dx}(x^{n+1}) = \frac{1}{n+1}((n+1)x^{n+1-1}) = x^n.$$

   Since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, we know an antiderivative of x^n is indeed $\frac{x^{n+1}}{n+1}$.
   So, an antiderivative of $f(x)$ is

   $$F(x) = \frac{x^{2+1}}{2+1} - 2\left(\frac{x^{1+1}}{1+1}\right) + 3\left(\frac{x^{0+1}}{0+1}\right) = \frac{1}{3}x^3 - x^2 + 3x.$$

   I can check this two ways: either graphically or by differentiating $F(x)$ to see if I get $f(x)$.
   Check:

   $$F'(x) = \frac{1}{3}(3x^2) - 2x + 3 = x^2 - 2x + 3 = f(x).$$

(b) $f(x) = x^3 - \dfrac{5}{x^2} + 2$

$$\begin{aligned} f(x) &= x^3 - 5x^{-2} + 2x^0 \\ \Rightarrow F(x) &= \frac{x^4}{4} - 5\left(\frac{x^{-1}}{-1}\right) + 2\left(\frac{x^1}{1}\right) \\ &= \frac{1}{4}x^4 + \frac{5}{x} + 2x \end{aligned}$$

Check:

$$F'(x) = \frac{1}{4}(4x^3) + 5(-1x^{-2}) + 2 = x^3 - \frac{5}{x^2} + 2 = f(x).$$

(c) $f(x) = x^\pi + x^{-42} - 17x$

$$F(x) = \frac{x^{\pi+1}}{\pi+1} + \frac{x^{-42+1}}{-42+1} - 17\left(\frac{x^2}{2}\right) = \frac{1}{\pi+1}x^{\pi+1} - \frac{1}{41x^{41}} - \frac{17}{2}x^2.$$

Check:

$$F'(x) = \frac{1}{\pi+1}\left((\pi+1)x^\pi\right) - \frac{1}{41}(-41x^{-42}) - \frac{17}{2}(2x) = x^\pi + x^{-42} - 17 = f(x).$$

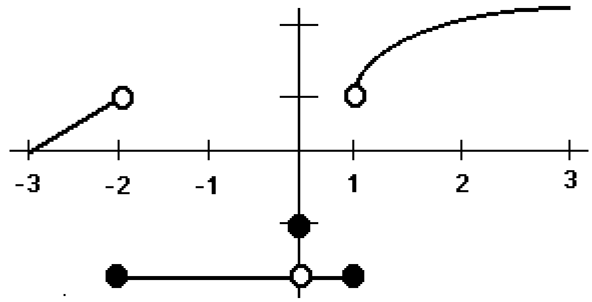
(d) $f(x) = 7x^{-1} - x + 4$

$$F(x) = 7\left(\frac{x^{-1+1}}{-1+1}\right) - \frac{x^2}{2} + 4x = \mathbf{?} - \frac{x^2}{2} + 4x.$$

Uh oh! Apparently, we can't antidifferentiate x^{-1} using the formula $\dfrac{x^{n+1}}{n+1}$!. That doesn't necessarily mean *no* function has x^{-1} as its derivative, but it certainly means no *power* function has x^{-1} as its derivative.

So for now, we're left wondering ... what's the deal with $1/x$???

Part II: The graph of a function f is shown below:



The Graph of f

1. Find the following limits:
   (a) $\lim_{x \to -1} f(x)$
   To find the limit as x approaches -1, the key is *not* to pay any attention at all to what f does *at* $x = -1$. Instead, we look to see what f does *near* $x = -1$.
   Since the points on the graph of $f(x)$ have coordinates $(x, f(x))$, asking what $f(x)$ does near $x = -1$ is the same as asking how the y-coordinates of the graph behave as x gets closer and closer to $x = -1$.
   In order to do this, we need to see what happens as x approaches -1 from both the left and the right.
   Mentally put a pencil on the graph of $f(x)$ at a point where the x-value is near to, but less than $x = -1$. $(-2, f(-2))$ will do. Now move the pencil point along the graph toward $x = -1$ (without actually touching $x = -1$). Are the y-values approaching any particular number? Yes they are – in this case, they're not changing at all, in fact. As x gets closer and closer to -1 from the left, the y-values are staying at what we will call $y = -2$. Because we were finding what $f(x)$ approached as x approached -1 from the left, or negative side, we write this as
   $$\lim_{x \to -1^-} f(x) = -2.$$
   Next we do the same thing, but beginning at a point on the graph where the x-value is near to, but *greater than* $x = -1$. This time $(-.5, f(-.5))$ might be a good choice. We move the mental pencil point along the graph toward $x = -1$ again,

paying attention to what the y-values are doing. Again, they stay at -2. We write this as

$$\lim_{x\to -1^+} f(x) = -2.$$

So, to find what $\lim_{x\to -1} f(x)$ is (if it exists), we need to put these two pieces of information together. *If* it doesn't matter whether we approach from the left or the right – that is, if $f(x)$ approaches the same value from both sides – then we know the limit is that value.
So, since the limit from the left and the limit from the right were both -2, we know

$$\lim_{x\to -1} f(x) = -2.$$

(b) $f(-1)$
To find $f(-1)$, we do the same thing we've always done. We don't look to either side of $x = -1$, we just mentally draw a vertical line from where the mark for $x = -1$ is on the x-axis to the graph, and the y coordinate of that intersection point is the y coordinate of $f(x)$.
In this case, this is very easy to do – $f(-1) = -2$.

(c) Is f continuous at $x = -1$?
In order for f to be continuous at $x = -1$, we need for $\lim_{x\to -1} f(x) = f(-1)$. We know that $\lim_{x\to -1} f(x) = -2$ and we know that $f(-1) = -2$. Since they are equal, $f(x)$ is indeed continuous at $x = -1$.

(d) $\lim_{x\to 0} f(x)$
Again, we don't pay any attention to what f is doing *at* $x = 0$. Instead, we look a bit to the left and a bit to the right. If follow the graph from about the point $(-1, f(-1))$ toward $x = 0$, we see that again $f(x) \to -2$. So

$$\lim_{x\to 0^-} f(x) = -2.$$

If we follow the graph from about the point $(1, f(1))$ toward $x = 0$, we also see that $f(x) \to -2$. So

$$\lim_{x\to 0^+} f(x) = -2.$$

Because $f(x)$ approaches -2 whether we come from the right or the left, we can conclude that

$$\lim_{x\to 0} f(x) = -2.$$

(e) $f(0)$
Again, we just look directly beneath $x = 0$, without glancing left or right, to see what the y-value of the graph is. This is perhaps a bit trickier – there's a hole in the horizontal line going through $y = -2$, so $f(0)$ is *not* -2.
Does that mean it's undefined? We look some more, and we see the solid dot at $(0, -1)$. This indicates that the function is defined to be -1 at $x = 0$. In other words,

$$f(0) = -1.$$

(f) Is f continuous at $x = 0$?
In order for f to be continuous at $x = 0$, we need for $\lim_{x\to 0} f(x) = f(0)$. We know that $\lim_{x\to 0} f(x) = -2$ and $f(0) = -1$. Since these are *not* equal, $f(x)$ is *not* continuous at $x = 0$.

(g) $\lim_{x\to -2} f(x)$
Looking first at what happens as x approaches -2 from the left, we see that the y-values are moving up from the x-axis toward $y = 1$, so we know

$$\lim_{x\to -2^-} f(x) = 1.$$

Now looking at what happens as x approaches -2 from the right, we see that the y-values are moving along the horizontal line, staying at $y = -2$, so we know

$$\lim_{x\to -2^+} f(x) = -2.$$

Because $f(x)$ does *not* approach the same number if we move along the graph from the left as if we move along from the right, we say

$$\lim_{x\to -2} f(x) \text{ does not exist.}$$

(h) $f(-2)$
Looking directly above and below $x = -2$, we see a hollow dot at $y = 1$ (which in essence *means* $f(x)$ approaches 1 as x approaches -2 from the left, *but* that $f(-2) \neq 1$. We also see a solid dot at $y = -2$, which means that $f(-2) = -2$.
So $f(-2) = -2$.

(i) Is f continuous at $x = -2$?
We would need the value of $\lim_{x\to -2} f(x)$ to equal the value of $f(-2)$, in order for f to be continuous there. Because the limit doesn't even exist, this obviously can not happen, and so $f(x)$ is *not* continuous at $x = -2$.

(j) $\lim_{x\to 1^+} f(x)$
$\lim_{x\to 1^+} f(x) = 1$.

(k) $\lim_{x\to 1^-} f(x)$
$\lim_{x\to 1^-} f(x) = -2$.

(l) $f(1)$
$f(1) = -2$.

(m) Is f continuous at $x = 1$?
Once again, because $\lim_{x\to 1} f(x)$ doesn't even exist, it can't equal $f(1)$, and so the function is not continuous there.

2. On which intervals is f continuous?

In practice, the definition of continuous agrees with your instinct for what continuity is. If you have to pick up your pencil then left and right sided limits are different, and so there's a point of discontinuity. Thus we can fairly quickly and easily decide the intervals on which f is continuous.

I can trace along f from $x = -3$ up to (but not including) $x = -2$ without lifting my pencil. Then at $x = -2$ I lift up the pencil, move it down, and trace along from $x = -2$ to $x = 0$, etc.

f is continuous on the intervals $[-3, -2) \cup [-2, 0) \cup (0, 1] \cup (1, 3]$.