Recall: The derivative function

Definition: The derivative function for $f(x)$ is the function:

- which gives the slope of the line tangent to f at each point $(x, f(x))$, if that slope exists.
- or equivalently which gives the instantaneous rate that f changes at each point $(x, f(x))$, if such a rate exists.

The derivative function is denoted $f'(x)$ and is read as *the derivative of $f(x)$*.

At every point $(x, f(x))$ where the function $f(x)$ has a tangent line, the derivative function has a point $(x, f'(x))$.

Recall: Formulas for the derivatives of two types of functions

- Since the graph of $f(x) = k$, where k is any constant, is a horizontal line (with slope 0 at every point), the slope of the tangent line (or the rate the function changes) is 0.

$$\text{If } f(x) = k, \text{ then } f'(x) = 0 \text{ for all } x$$

- Since the graph of $f(x) = mx + b$, where m and b are any constants, is a line with slope m, the slope of the tangent line (or the rate the function changes) is m.

$$\text{If } f(x) = mx + b, \text{ then } f'(x) = m \text{ for all } x$$

Recall: The limit definition of the derivative of a function at a point $(a, f(a))$

If $f(x)$ is a function that has a tangent line at the point $(a, f(a))$, then the slope of that tangent line is given by

$$f'(a) = \lim_{h \to \infty} \frac{f(a+h) - f(a)}{h}.$$

Recall that the difference quotient $\frac{f(a+h) - f(a)}{h}$ gives the slope of the secant line between the point $(a, f(a))$ and a point h horizontal units away, $(a+h, f(a+h))$, and equivalently, also gives the average rate of change of f between those two points. As the horizontal distance between the two points gets smaller, the secant line gets closer to the tangent line; the average rate of change gets closer to the instantaneous rate of change.

Recall: Tangent lines; Shifting vertically

We found

- If $f(x) = x^2$, then by using the limit definition we could find $f'(1) = 2$
- The the line tangent to $f(x)$ at the point $(1, 1)$ has slope 2
- Using the point-slope equation of a line $y - y_1 = m(x - x_1)$, we found that the line tangent to $f(x)$ at $x = 1$ has the equation

$$y - 1 = 2(x - 1) \text{ or } y = 2x - 1.$$

- Also realized: For all functions $f(x) = x^2 + k$ for any constant k, the slope at $x = 1$ is 2

DWW due 2/23 Problem 1

Let $f(x) = x^3 - 12x$. Calculate diff quotient $\dfrac{f(3+h) - f(3)}{h}$ for

(a) $h = 0.1$: $\dfrac{\left[(3+0.1)^3 - 12(3+0.1)\right] - \left[3^3 - 12(3)\right]}{0.1} = 15.91$

(b) $h = 0.01$: $\dfrac{\left[(3+0.01)^3 - 12(3+0.01)\right] - \left[3^3 - 12(3)\right]}{0.01} = 15.0909999$

(c) $h = -0.01$:
$\dfrac{\left[(3-0.01)^3 - 12(3-0.01)\right] - \left[3^3 - 12(3)\right]}{-0.01} = 14.910100$

(d) $h = -0.1$: $\dfrac{\left[(3-0.1)^3 - 12(3-0.1)\right] - \left[3^3 - 12(3)\right]}{-0.1} = 14.11$

If someone now told you that the derivative (slope of the tangent line to the graph of $f(x)$) at $x = 3$ was an integer, what would you expect it to be?

$f'(3) = \lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$. As $h \to 0$ from the right or left, it seems that $\dfrac{f(3+h) - f(3)}{h} \to 15$, so I would expect $f'(3) = 15$.

DWW due 2/23 Problem 4

$\lim_{h\to 0} \dfrac{(6+h)^3 - 216}{h}$ represents a derivative $f'(a)$. Find $f(x)$ and a.

$$f'(a) \stackrel{\text{def}}{=} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{(6+h)^3 - 216}{h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Reasonable start: $f(a+h) = (6+h)^3$ and $f(a) = 216$.
From $f(a+h) = (6+h)^3$, it seems $a = 6$ and $f(x) = x^3$.
If so, would $f(a) = 216$, as needed? $6^3 = 216$, so yes ✓
Check: If $f(x) = x^3$ and $a = 6$, do we get the given derivative?

$$f'(a) \stackrel{\text{def}}{=} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(6+h)-f(6)}{h} = \lim_{h\to 0} \frac{(6+h)^3 - 6^3}{h} \checkmark$$

Generalizing:

Finding a formula for the $\left\{ \begin{array}{c} \text{slope of the tangent line} \\ \text{instantaneous rate of change} \\ \text{derivative} \end{array} \right\}$ of $f(x)$ at any point $(x, f(x))$

Generalizing:

Finding a formula for the $\left\{ \begin{array}{c} \text{slope of the tangent line} \\ \text{instantaneous rate of change} \\ \text{derivative} \end{array} \right\}$ of $f(x)$ at any point $(x, f(x))$

- To find the slope/inst. r.o.c./derivative of $f(x)$ at a specific point $(a, (f(a))$, use the limit definition of the derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Generalizing:

Finding a formula for the $\left\{ \begin{array}{c} \text{slope of the tangent line} \\ \text{instantaneous rate of change} \\ \text{derivative} \end{array} \right\}$ of $f(x)$ at any point $(x, f(x))$

- To find the slope/inst. r.o.c./derivative of $f(x)$ at a specific point $(a, (f(a))$, use the limit definition of the derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

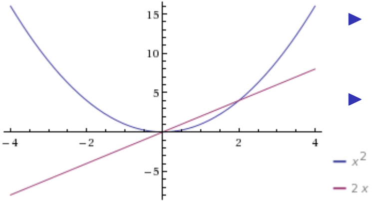
- To find slope/inst. r.o.c./derivative of $f(x)$ at any point $(x, f(x))$, use an almost identical definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Making sense of the result: when $f(x) = x^2$, $f'(x) = 2x$

- Where does $f \uparrow$, and where does $f \downarrow$?

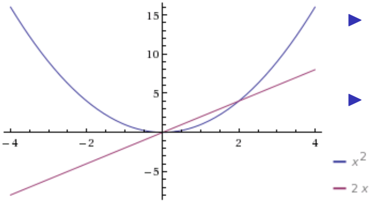
- Where does f have slope > 0? < 0?



- Where is $f' > 0$? $f' < 0$?

- What more can we say about the rate f changes?

Making sense of the result: when $f(x) = x^2$, $f'(x) = 2x$



- Where does $f \uparrow$, and where does $f \downarrow$?
  $f \uparrow$ on $(0, \infty)$ and $\downarrow$ on $(-\infty, 0)$.
- Where does f have slope > 0? < 0?
  Slope of f is > 0 on $(0, \infty)$ & < 0 on $(-\infty, 0)$.
- Where is $f' > 0$? $f' < 0$?
  $f' > 0$ on $(0, \infty)$ and $f' < 0$ on $(-\infty, 0)$.
- What more can we say about the rate f changes?

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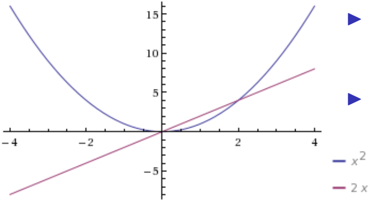
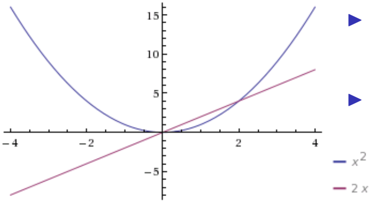


- Where does $f \uparrow$, and where does $f \downarrow$?
  $f \uparrow$ on $(0, \infty)$ and $\downarrow$ on $(-\infty, 0)$.
- Where does f have slope > 0? < 0?
  Slope of f is > 0 on $(0, \infty)$ & < 0 on $(-\infty, 0)$.
- Where is $f' > 0$? $f' < 0$?
  $f' > 0$ on $(0, \infty)$ and $f' < 0$ on $(-\infty, 0)$.
- What more can we say about the rate f changes?
  On $(0, \infty)$, f increases faster and faster. That is, the r.o.c. of f is increasing.

Making sense of the result: when $f(x) = x^2$, $f'(x) = 2x$



- Where does $f \uparrow$, and where does $f \downarrow$?
  $f \uparrow$ on $(0, \infty)$ and $\downarrow$ on $(-\infty, 0)$.
- Where does f have slope > 0? < 0?
  Slope of f is > 0 on $(0, \infty)$ & < 0 on $(-\infty, 0)$.
- Where is $f' > 0$? $f' < 0$?
  $f' > 0$ on $(0, \infty)$ and $f' < 0$ on $(-\infty, 0)$.
- What more can we say about the rate f changes?
  On $(0, \infty)$, f increases faster and faster.
  That is, the r.o.c. of f is increasing.

  On $(-\infty, 0)$, $f \downarrow$ slower and slower.
  That is, the r.o.c. of f goes from being very negative to being closer to 0.

In Class Work

1. Use the limit definition to find the slope of the line tangent to $f(x)$ at the variable point x:
   (a) $f(x) = x^2 - 3x$
   (b) $f(x) = \frac{1}{x} = x^{-1}$ *Hint:* In the numerator, find a common denominator. Simplify, then simplify some more.
   (c) $f(x) = \sqrt{x} = x^{1/2}$ *Hint:* Multiply by 1 in a way that will eliminate the square roots in the numerator.
2. Find the second derivative $f''(x)$ for the function from 1(a), $f(x) = x^2 - 3x$.

Solutions

1(a) Use the limit definition to find the slope of the line tangent to $f(x) = x^2 - 3x$ at the variable point x

$$
\begin{aligned}
m_{tan} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\Big[(x+h)^2 - 3(x+h)\Big] - \Big[x^2 - 3(x)\Big]}{h} \\
&= \lim_{h\to 0} \frac{\left(x^2 + 2xh + h^2 - 3x - 3h\right) - \left(x^2 - 3x\right)}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\
&= \lim_{h\to 0} 2x + h - 3 \\
&= 2x - 3
\end{aligned}
$$

Solutions

1(b) Use the limit definition to find the slope of the line tangent to $f(x) = \frac{1}{x} = x^{-1}$ at the variable point x

$$
\begin{aligned}
m_{tan} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{x+h}\right) - \left(\frac{1}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x\cdot 1}{x(x+h)} - \frac{1\cdot x+h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{\frac{h}{1}} = \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} = -x^{-2}
\end{aligned}
$$

Solutions

1(c) Use the limit definition to find the slope of the line tangent to $f(x) = \sqrt{x} = x^{1/2}$ at the variable point x:

$$
\begin{aligned}
m_{tan} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \mathrm{dim}_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}
\end{aligned}
$$

Solutions

2. Find the second derivative $f''(x)$ for the function from 1(a), $f(x) = x^2 - 3x$.

$$f''(x) = \big(f'(x)\big)' = \big(2x - 3\big)'.$$

Since the graph of $y = 2x - 3$ is a line with slope 2,

$$f''(x) = \big(f'(x)\big)' = \big(2x - 3\big) = 2$$

Observations So Far: Looking for Patterns

- If $f(x) = k$, $f'(x) = 0$.
  - If $f(x) = 1 = x^0$, $f'(x) = 0$
  - If $g(x) = k = k \cdot 1$, $g'(x) = 0 = k \cdot 0$
- If $f(x) = mx + b$, $f'(x) = m$.
  - If $f(x) = x = x^1$, $f'(x) = 1$
  - If $g(x) = kx = k \cdot x$, $g'(x) = k = k \cdot 1$
  - If $h(x) = kx + c$, $h'(x) = k$
- If $f(x) = x^2$, $f'(x) = 2x$
  - If $h(x) = x^2 + k$, $h'(x) = 2x$
- If $f(x) = \dfrac{1}{x} = x^{-1}$, $f'(x) = -\dfrac{1}{x^2} = -x^{-2}$
- If $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$