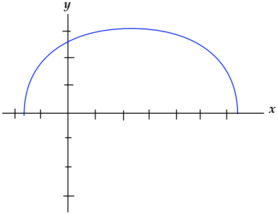
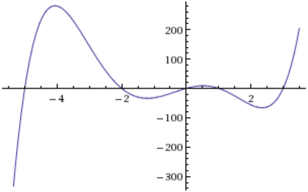
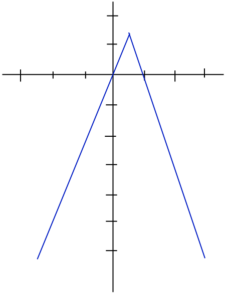
Polynomials

Question: Which of the following *could* be the graph of a polynomial?

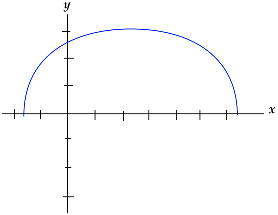






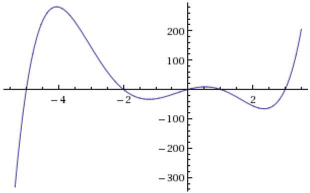
Polynomials

Solution:



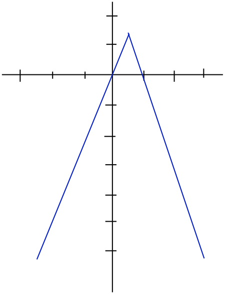
Domain is not all of $\mathbb{R}$

Not a polynomial



Domain could be all of $\mathbb{R}$
Graph is smooth, unbroken
No asymptotes

Could be a polynomial



Graph is not smooth

Not a polynomial

Fractional Powers

Fractional powers represent roots:

- To see that $\sqrt{a} = a^{\frac{1}{2}}$:

$$\begin{aligned} a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} &= a \\ &\textit{and} \\ \sqrt{a} \cdot \sqrt{a} &= a \end{aligned}$$

- To see that $\sqrt[4]{a} = a^{\frac{1}{4}}$

$$\begin{aligned} a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} + a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} = a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} &= a \\ &\textit{and} \\ \sqrt[4]{a} \cdot \sqrt[4]{a} \cdot \sqrt[4]{a} \cdot \sqrt[4]{a} &= a \end{aligned}$$

Fractional Powers

In general:

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$
- Domain and range of $f(x) = x^{\frac{1}{n}}$:
  - If n is even, the domain and range of $f(x) = x^{\frac{1}{n}}$ are both $[0, \infty)$
  - If n is odd, the domain and range of $f(x) = x^{\frac{1}{n}}$ are both $(-\infty, \infty)$ (but many graphing technologies will only graph in first quadrant).

Question:

What are the domain and range of $f(x) = (16 - x)^{3/4}$?

Fractional Powers

Solution:

$$f(x) = (16 - x)^{3/4} = \left((16 - x)^{\frac{1}{4}}\right)^3$$

- When the power is reduced as far as possible, the denominator is even.
- Domain: $16 - x \geq 0 \implies x \leq 16 \implies$ Domain: $x \in (-\infty, 16]$
- Range: $(16 - x)^{\frac{1}{4}} \geq 0 \implies \left((16 - x)^{\frac{1}{4}}\right)^3 \implies$ Range: $y \in [0, \infty)$

Negative Powers

Negative powers represent fractions:

- **Remember:** $a^0 = 1$ for any real a
- To see that $\frac{1}{a^n} = a^{-n}$, for $n \in \mathbb{R}$

$$\begin{aligned} a^{-n} \cdot a^n = a^0 &= 1 \\ &\textit{and} \\ \frac{1}{a^n} \cdot a^n &= 1 \end{aligned}$$

- Assuming n is a positive integer:
  Domain and range of $f(x) = x^{-n}$:
  - Since $x^{-n} = \frac{1}{x^n}$, the domain is $(-\infty, 0) \bigcup (0, \infty)$
  - Since there is no value of x for which $\frac{1}{x^n} = 0$, the range is
    - $(-\infty, 0) \bigcup (0, \infty)$, if n is odd
    - $(0, \infty)$ if n is even

Negative Powers

Question:
What are the domain and range of $f(x) = (16 - x)^{-3/4}$?

Negative Powers

Solution:

$$f(x) = (16-x)^{-3/4} = \frac{1}{\left((16-x)^{\frac{1}{4}}\right)^3}$$

- We already found the domain and range of the denominator:
  - Domain of denominator: $x \in (-\infty, 16]$
  - Range of denominator: $y \in [0, \infty)$
- The only new thing to consider is that because in $f(x) = (16-x)^{-3/4}$, $(16-x)^{\frac{1}{4}}\big)^3$ is in the denominator, the part we're raising to the 3/4 power can't be 0, nor can 0 come out of it.
  - Domain of $f(x) = (16-x)^{-3/4}$: $x \in (-\infty, 16)$
  - Range of denominator: $y \in (0, \infty)$

Rational Functions

Rational functions are functions which can be written as the quotient of two polynomials

Example: $f(x) = \dfrac{4x^2 + 8x - 12}{x^2 - x - 6} = \dfrac{4(x+3)(x-1)}{(x-3)(x+2)}$

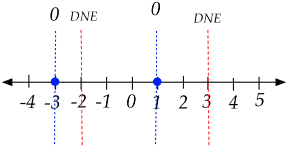
- Roots: $f(x) = 0$ where numerator is 0
  $\Rightarrow f(x)$ has roots at $x = -3$, $x = 1$
- Vertical Asymptotes: $f(x)$ has vertical asymptotes where denominator is 0 (after all factoring and simplification has been done).
  $\Rightarrow f(x)$ has vertical asymptotes at $x = 3$ and $x = -2$.
- Horizontal Asymptotes:
  - Because the numerator & denominator have same degree (both have degree 2), f has a horizontal asymptote.
  - The horizontal asymptote is at $y = \frac{4}{1}$, the ratio of the leading coefficients of numerator and denominator

Rational Functions

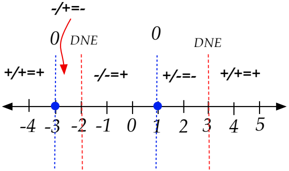
Example (continued)

Sketch $f(x) = \dfrac{4(x+3)(x-1)}{(x-3)(x+2)}$

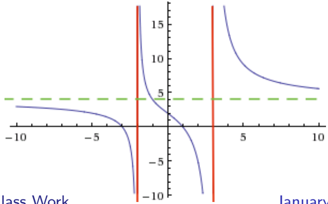
Mark the zeroes and vertical asymptotes:



These are the only places where $f(x)$ can change sign, so pick one point in each interval at which to evaluate the sign



Put on a set of axes, add in the horizontal asymptote (if there is one), and sketch



Absolute Value Functions

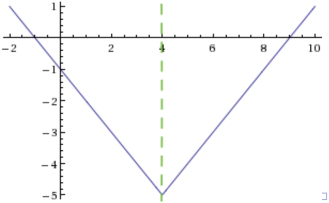
Question: Sketch $f(x) = |x - 4| - 5$

Absolute Value Functions

$$f(x) = \begin{cases} (x-4)-5 & \text{if } x-4 \geq 0 \\ -(x-4)-5 & \text{if } x-4 < 0 \end{cases}$$

$$f(x) = \begin{cases} x-9 & \text{if } x \geq 4 \\ -x-1 & \text{if } x < 4 \end{cases}$$

Thus to the right of $x = 4$, the graph is the line $y = x - 9$, while to the left of $x = 4$, the graph is the line $y = -x - 1$. Note that these two lines intersect at the point $(4, -5)$.



In Class Work

1. Find the domain and zeroes of $f(x) = (x^{-2}+1)^{\frac{1}{4}}$
2. Without using a calculator or graphing utility, sketch the graph of $f(x) = \dfrac{2x^2-2x}{x^2-4}$, which factors to $f(x) = \dfrac{2x(x-1)}{(x-2)(x+2)}$ by finding where $f(x)$ has zeroes, horizontal asymptotes, vertical asymptotes, and by doing a sign analysis.

Solutions:

1. Find the domain and zeroes of $f(x) = (x^{-2} + 1)^{\frac{1}{4}}$

   - Domain: 4 even $\implies x^{-2} + 1$ must be non-negative.

     $$x^{-2} + 1 \geq 0 \Rightarrow \frac{1}{x^2} \geq -1 \qquad \text{True for all } x$$

     Domain: $x \in (-\infty, \infty)$

   - Zeroes: $f(x) = 0 \Leftrightarrow \frac{1}{x^2} + 1 = 0.$
     Since $\frac{1}{x^2} + 1 \neq 0$, $f(x)$ has no zeroes

Solutions:

2. Without using a calculator or graphing utility, sketch the graph of $f(x) = \frac{2x^2 - 2x}{x^2 - 4}$, which factors to $f(x) = \frac{2x(x-1)}{(x-2)(x+2)}$ by finding where $f(x)$ has zeroes, horizontal asymptotes, vertical asymptotes, and by doing a sign analysis.

   - Zeroes: $f(x) = 0$ where numerator$=0$ (after simplifying)
     $\Longrightarrow$ Zeroes: $x = 0$, $x = 1$
   - Vertical Asymptotes: $f(x)$ has vertical asymptotes where denom$=0$ (after simplifying)
     $\Longrightarrow$ Vertical Asymptotes: $x = 2$, $x = -2$
   - Horizontal Asymptotes: $f(x)$ has a horizontal asymptotes **if** (after simplifying) the degree of the numerator and denominator are the same. Degree of each$=2$, so $f(x)$ *has* a horizontal asymptote

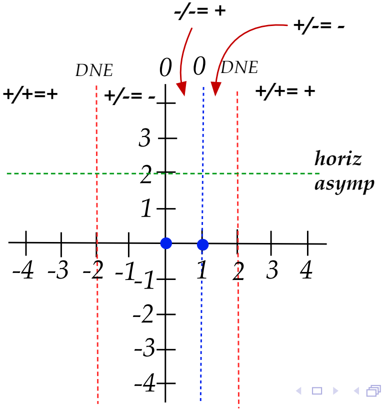
     Horizontal asymptote $y =$ ratio of leading coefficients $= \frac{2}{1}$
     $\Longrightarrow$ Horizontal Asymptote: $y = 2$

Solutions:

2. Sketch the graph of $f(x) = \dfrac{2x^2 - 2x}{x^2 - 4} = \dfrac{2x(x-1)}{(x-2)(x+2)}$.

   - Zeroes: $x = 0$, $x = 1$
   - Vertical Asymptotes: $x = 2$, $x = -2$
   - Horizontal Asymptotes: $y = 2$

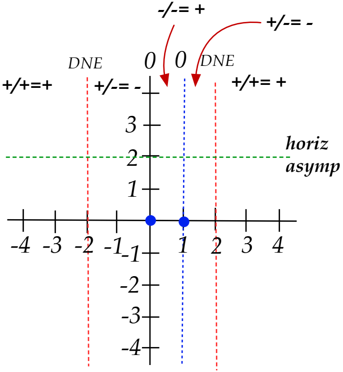
Sign Analysis:



Solutions:

2. Sketch the graph of $f(x) = \frac{2x^2 - 2x}{x^2 - 4} = \frac{2x(x-1)}{(x-2)(x+2)}$.

Sign Analysis:



Sketch: