Daily WeBWorK: Let $f(x) = (x^2 - 16)^4$

- $f'(x) = 4(x^2 - 16)^3 \cdot 2x = 8x(x^2 - 16)^3 \Rightarrow f'(x) = 0$ when $x = 0, \pm 4$. Since $f'(x)$ exists everywhere, Critical Points: $\{-4, 0, 4\}$.
- Increasing/decreasing?
  - $f'(-5) = 4(+)(-) < 0 \Rightarrow f \downarrow$ on $(-\infty, -4)$
  - $f'(-1) = 4(-)(-) > 0 \Rightarrow f \uparrow$ on $(-4, 0)$
  - $f'(1) = 4(-)(+) < 0 \Rightarrow f \downarrow$ on $(0, 4)$
  - $f(5) = 4(+)(+) > 0 \Rightarrow f \uparrow$ on $(4, \infty)$

  f has a local max at $x = 0$, and local mins at $x = \pm 4$.
- Inflection Points?

$$\begin{aligned} f''(x) &= 8(x^2 - 16)^3 + 8x \cdot 3(x^2 - 16)^2 \cdot 2x = 8(x^2 - 16)^3 + 48x^2(x^2 - 16 \\ &= 8(x^2 - 16)^2\left[(x^2 - 16) + 6x^2\right] = 8(x^2 - 16)^2(7x^2 - 16) \end{aligned}$$

Possible Inflection Points: $x = \pm 4, x = \pm \frac{4}{\sqrt{7}}$, but since $x = \pm 4$ are local mins and $f(x)$ is smooth, they are not inflection points.

$f''(10) = (+)(+) > 0$, $f''(0) = (+)(-) < 0$, $f''(10) = (+)(+) > 0 \Rightarrow$ f is concave up on $\left(-\infty, -\frac{4}{\sqrt{7}}\right) \cup \left(\frac{4}{\sqrt{7}}, \infty\right)$ and concave down on $\left(-\frac{4}{\sqrt{7}}, \frac{4}{\sqrt{7}}\right)$

Recall: Extreme Value Theorem (Chapter 1)

If f is continuous on a closed interval $[a, b]$, then there exist values M and m in the interval $[a, b]$ such that $f(M)$ is the maximum value of $f(x)$ on $[a,]$ and $f(m)$ is the minimum value of $f(x)$ on $[a, b]$.

Application of Differentiation: Optimization

People need to find when and where maxima and minima occur all the time:

- Maximum revenue
- Minimum cost
- Maximum speed
- Minimum population
- Minimum surface area for a certain volume
- Maximum amount of a drug in the bloodstream
- Maximum yearly temperature

Definition: Absolute/Global Extrema

Definition: On an interval I,

- f has an **absolute** or **global maximum** at $x = c$, and $f(c)$ is the **absolute** or **global maximum value** for f on I if the point $(c, f(c))$ is as high or higher than any other point on the graph of f over the interval I. That is, if $f(c) \geq f(x)$ for all $x \in I$.

- f has an **absolute** or **global minimum** at $x = c$, and $f(c)$ is an **absolute** or **global minimum value** for f on I if the point $(c, f(c))$ is as low or lower than any other point on the graph of f over the interval I.

In Class Work

Find the absolute extrema of the given function on each indicated interval. If you have a graphing calculator available, check your results by looking at a graph.

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$
2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1, 1]$ and (b) $[-1, 8]$

Solutions:

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$

(a) $[-3, 1]$

- **Critical Points**: $f'(x) = 4x^3 - 16x$.
  - f' exists everywhere
  - $$\begin{aligned} f'(x) = 0 \quad &\implies \quad 0 = 4x^3 - 16x \\ &\implies \quad x^3 - 4x = 0 \\ &\implies \quad x(x^2 - 4) = 0 \\ &\implies \quad x = 0, x = 2, x = -2 \end{aligned}$$

  The Critical Points are $x = -2$, $x = 0$, and $x = 2$.

Solutions:

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$

(a) $[-3, 1]$

- **Critical Points:** $x = 0, x = 2, x = -2$. Only -2 and 0 lie in this interval.
- **Compute f itself at endpoints and critical points**:
  - $f(-3) = (-3)^4 - 8(-3)^2 + 2 = 81 - 72 + 2 = 11$
  - $f(-2) = (-2)^4 - 8(-2)^2 + 2 = 16 - 32 + 2 = -14$
  - $f(0) = 0^4 - 8(0)^2 + 2 = 2$
  - $f(1) = (1)^4 - 8(1)^2 + 2 = 1 - 8 + 2 = -5$
- **Compare:**
  On the interval $[-3, 1]$, f attains an absolute maximum value of $y = 11$ at $x = -3$ and an absolute mininum value of $y = -14$ at $x = -2$.

Solutions:

1. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$

(b) $[-1, 3]$

- **Critical Points:** Same as in (a): $x = -2, 0, 2$. Only 0 and 2 lie in this interval.
- **Compute f itself at endpoints and critical points**
  - $f(-1) = (-1)^4 - 8(-1)^2 + 2 = 1 - 8 + 2 = -5$
  - $f(0) = 2$
  - $f(2) = 2^4 - 8(2)^2 + 2 = -14$
  - $f(3) = 3^4 - 8(3)^2 + 2 = 11$
- **Compare:**
  On $[-1, 3]$, f attains an absolute maximum value of $y = 11$ at $x = 3$ and an absolute mininum value of $y = -14$ at $x = 2$.

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1, 1]$ and (b) $[-1, 8]$

(a) $[-1, 1]$

- **Critical Points**

$$
\begin{aligned}
f'(x) &= x^{2/3}\left[3(x-5)^2\right] + \frac{2}{3}x^{-1/3}(x-5)^3 \\
&= \frac{1}{x^{1/3}}\left(x^1\left[3(x-5)^2\right] + \frac{2}{3}(x-5)^3\right) \\
&= \frac{(x-5)^2}{x^{1/3}}\left(3x + \frac{2}{3}(x-5)\right) \\
&= \frac{(x-5)^2}{x^{1/3}}\left(3x + \frac{2}{3}x - \frac{10}{3}\right) \\
&= \frac{(x-5)^2\left(\frac{11}{3}x - \frac{10}{3}\right)}{x^{1/3}}
\end{aligned}
$$

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1,1]$ and (b) $[-1,8]$

(a) $[-1,1]$

- **Critical Points**
  - $f'(x) = \dfrac{(x-5)^2\left(\frac{11}{3}x - \frac{10}{3}\right)}{x^{1/3}}$
  - f' does not exist at $x=0$, but f does
  - $f' = 0$ when $x = 5$ or when $\dfrac{11}{3}x = \dfrac{10}{3}$, that is when $x = \dfrac{10}{11}$.
  - Thus the Critical Points are $x = 0, \frac{10}{11}, 5$. Only $x = 0$ and $x = 10/11$ lie in this interval.

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1,1]$ and (b) $[-1,8]$

(a) $[-1,1]$

- **Critical Points**: $x = 0, \frac{10}{11}, 5$. Only $x = 0$ and $x = 10/11$ lie in $[-1,1]$.
- **Compute**
  - $f(-1) = ((-1)^2)^{1/3}(-1-5)^3 = (-6)^3 = -216$
  - $f(0) = (0)^{2/3}(0-5)^3 = 0$
  - Using a calculator, $f(10/11) \approx -64.25$
  - $f(1) = 1^{2/3}(1-5)^3 = (-4)^3 = -64$
- **Compare**
  On $[-1,1]$, f has absolute maximum value of 0 at $x = 0$ and an an absolute mininum value of $y = -216$ at $x = -1$.

Solutions:

2. $f(x) = x^{2/3}(x-5)^3$ on (a) $[-1, 1]$ and (b) $[-1, 8]$

(b) $[-1, 8]$

- **Critical Points**: Still $x = 0, \frac{10}{11}, 5$; all lie in our interval.
- **Compute**
  - $f(-1) = -216$
  - $f(0) = 0$
  - $f(10/11) \approx -64.25$
  - $f(5) = (5)^{2/3}(5-5)^3 = 0$
  - $f(8) = (8)^{2/3}(8-5)^3 = (2)^2(3)^3 = (4)(27) = 108$
- **Compare**
  On $[-1, 8]$, f has absolute maximum value of 108 at $x = 8$ and an an absolute mininum value of -216 at $x = -1$.