What we know so far:

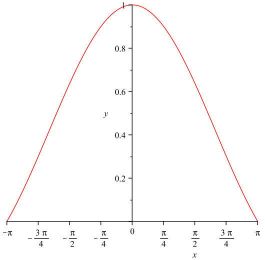
Function	Derivative
x^k	kx^{k-1}
e^x	e^x
b^x	$\ln(b)b^x$
$\ln(x)$	$\frac{1}{x}$
$\log_b(x)$	$\frac{1}{\ln(b)\cdot x}$

Type of fn	Derivative
$(fg)(x)$	$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
$\left(\frac{f}{g}\right)(x)$	$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$
$f\left(g(x)\right)(x)$	$f'\left(g(x)\right)g'(x)$

Types of Functions We Can't Yet Differentiate

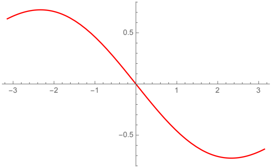
- $f(x) = (x^6 - 14x^5 + 27x^{-3} - 13)(101x^{-1} + 14x^6 + 13 - 42\sqrt{x})$ ✓
- $g(x) = \dfrac{x^7 - \sqrt{x}}{14x^2 + 12}$ ✓
- $h(x) = \left(x^2 + 1\right)^{25}$ ✓
- $j(x) = \cos(x^2)$
- $k(x) = \sin(e^{14x})$
- $m(x) = \ln(\sqrt{x} - 14)$ ✓

Recall Two Trig Limits



Graph of $\frac{\sin(h)}{h}$ on $[-\pi, \pi]$

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1$$



Graph of $\frac{\cos(h)-1}{h}$ on $[-\pi, \pi]$

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$$

Connection Between These Limits and Slopes

Note that both of these limits are derivatives at $x = 0$:

$$\frac{d}{dx}\big(\sin(x)\big)\Big|_{x=0} = \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h} = \lim_{h\to 0}\frac{\sin(h)}{h}$$

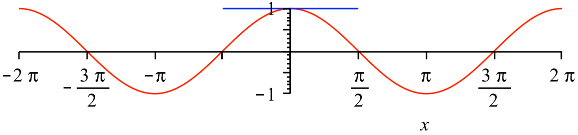
$$= \text{Slope of } \sin(x) \text{ at } x = 0$$

$$\frac{d}{dx}\big(\cos(x)\big)\Big|_{x=0} = \lim_{h\to 0}\frac{\cos(0+h)-\cos(0)}{h} = \lim_{h\to 0}\frac{\cos(h)-1}{h}$$

$$= \text{Slope of } \cos(x) \text{ at } x = 0$$

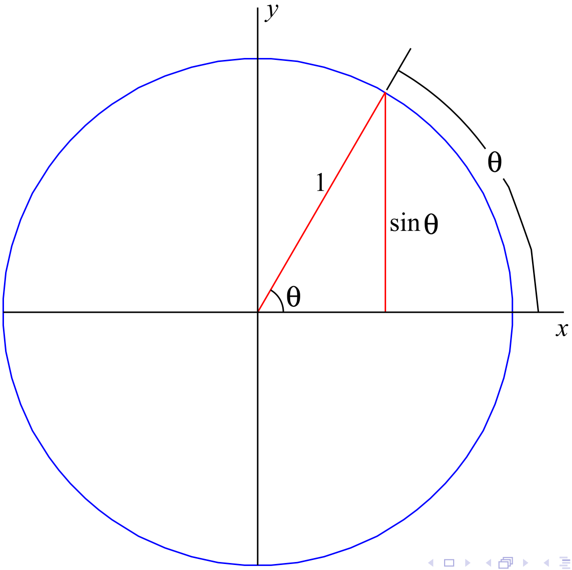
Alternative Reason Why $\lim_{h\to 0} \dfrac{\cos(h)-1}{h} = 0$:

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = \lim_{h\to 0} \frac{\cos(0+h)-\cos(0)}{h} = \frac{d}{dx}\big(\cos(x)\big)\Big|_0 .$$



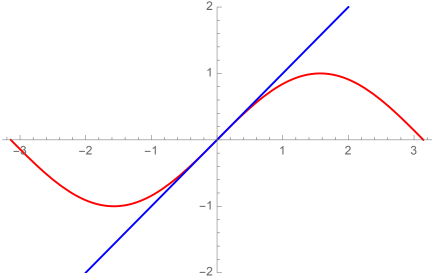
Since $\cos(x)$ clearly has a horizontal tangent line at $x = 0$, the slope at $x = 0$ is 0, so we know the derivative of $\cos(x)$ at $x = 0$ is 0, and hence this limit is 0.

Recall - radians



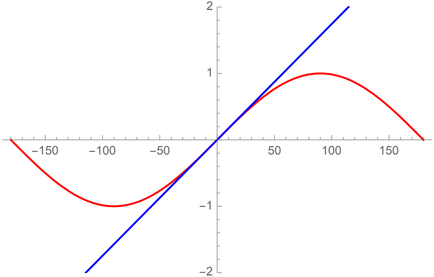
Why We Use Radians

Graph of $\sin(x)$ in radians



Tangent Line has slope 1

Graph of $\sin(x)$ in degrees



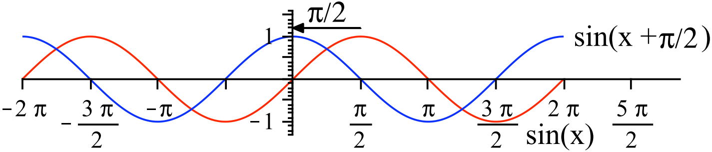
Tangent Line does not have slope 1
(It turns out to be $\frac{\pi}{180}$)

Recall:

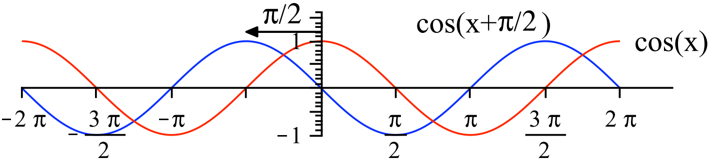
Trig identity:

$$\sin(x + h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Cosine and Sine are horizontal shifts of each other



$\cos(x) = \sin(x)$ shifted left $\frac{\pi}{2} = \sin\left(x + \frac{\pi}{2}\right)$



$-\sin(x) = \cos(x)$ shifted left $\frac{\pi}{2} = \cos\left(x + \pi/2\right)$

In Class Work

1. Find the derivatives of the following:
   (a) $f(x) = \tan(x)$ (Remember, $\tan(x) = \frac{\sin(x)}{\cos(x)}$)
   (b) $g(x) = \csc(x)$ (Remember, $\csc(x) = \frac{1}{\sin(x)}$)
   (c) $h(x) = \cos(e^x) + \sin(x+3)$
   (d) $j(x) = \sqrt{x}\sin(5x^2)$
   (e) $k(x) = \sec\big(\ln(x)\big) + 7$
2. Find an antiderivative of the following; check your answers by taking the derivative.
   (a) $f(x) = \cos(x) - \sin(x)$
   (b) $h(x) = 3\sin(4) + 2\sin(3x) + x^{732} + \frac{1}{x}$

Solutions

1. Find the derivatives of the following:

   (a) $f(x) = \tan(x)$ (Remember, $\tan(x) = \frac{\sin(x)}{\cos(x)}$)

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)(\cos(x)) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \left(\frac{1}{\cos(x)}\right)^2 \\
&= \sec^2(x)
\end{aligned}
$$

$$\boxed{\frac{d}{dx}\big(\tan(x)\big) = \sec^2(x)}$$

Solutions

1. Find the derivatives of the following:

   (b) $g(x) = \csc(x)$ (Remember, $\csc(x) = \frac{1}{\sin(x)}$)

$$
\begin{aligned}
g'(x) &= ddx\left(\big(\sin(x)\big)^{-1}\right) = -1\big(\sin(x)\big)^{-2}\big(\cos(x)\big) \\
&= -\frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)} \\
&= -\csc(x)\cot(x)
\end{aligned}
$$

$$\boxed{\frac{d}{dx}\big(\csc(x)\big) = -\csc(x)\cot(x)}$$

Solutions

1. Find the derivatives of the following:

   (c) $h(x) = \cos(e^x) + \sin(x+3)$

   Both $\cos(e^x)$ and $\sin(x+3)$ are compositions, so use the chain rule on each of them:

   $$h'(x) = -\sin(e^x)(e^x) + \cos(x+3)(1) = -27\sin(27x) + \cos(x+3).$$

   (d) $j(x) = \sqrt{x}\sin(5x^2)$

   Product rule, then chain rule when we get to it:

   $$\begin{aligned} j'(x) &= \sqrt{x}\frac{d}{dx}\Big(\sin(5x^2)\Big) + \frac{1}{2}x^{-1/2}\sin(5x^2) \\ &= \sqrt{x}\Big(\cos(5x^2)(10x)\Big) + \frac{\sin(5x^2)}{2\sqrt{x}} \end{aligned}$$

Solutions

1. Find the derivatives of the following:

   (e) $k(x) = \sec\big(\ln(x)\big) + 7$

   First part is a composition, with $\sec(u)$ being the outer function, $\ln(x)$ the inner.

$$k'(x) = \sec(u)\tan(u)\frac{d}{dx}\big(\ln(x)\big) = \sec\big(\ln(x)\big)\tan\big(\ln(x)\big)\cdot\frac{1}{x}$$

Solutions

2. Find an antiderivative of the following; check your answers by taking the derivative.

   (a) $f(x) = \cos(x) - \sin(x)$

   Since
   $$\frac{d}{dx}\big(\sin(x)\big) = \cos(x) \qquad \frac{d}{dx}\big(\cos(x)\big) = -\sin(x),$$
   an *antiderivative* of $\cos(x)$ is $\sin(x)$, and an *antiderivative* of $\sin(x)$ is $-\cos(x)$.

   Therefore, one antiderivative of $f(x)$ is:
   $$F(x) = \sin(x) - \big(-\sin(x)\big) = \sin(x) + \cos(x).$$

   Check: $F'(x) = \frac{d}{dx}\big(\sin(x) + \cos(x)\big) = \cos(x) - \sin(x) = f(x)$

Solutions

2. Find an antiderivative of the following; check your answers by taking the derivative.

   (b) $h(x) = 3\sin(4) + 2\sin(3x) + x^{732} + \frac{1}{x}$

   Piece-by-piece:

   - $3\sin(4)$: Just a constant. The antiderivative of a constant k is kx, so we have $3\sin(4)x$.
   - $2\sin(3x)$: Is an antiderivative $-2\cos(3x)$? Not quite - $\dfrac{d}{dx}\big(-2\cos(3x)\big) = -\big(-2\sin(3x)(3)\big) = 6\sin(3x)$. Off by a factor of 3. Try $-\frac{2}{3}\cos(3x)$ for the antiderivative. Then $\dfrac{d}{dx}\big(-\dfrac{2}{3}\cos(3x)\big) = 2\sin(3x)$, as we want.
   - x^{732}: An antiderivative is $\dfrac{1}{733}x^{733}$
   - $\frac{1}{x}$: Recall that $\dfrac{d}{dx}\big(\ln(x)\big) = \dfrac{1}{x}$, so an antiderivative of $\frac{1}{x}$ is $\ln(x)$.

   So an antiderivative is

$$H(x) = 3\sin(4)x - \frac{2}{3}\cos(3x) + \frac{1}{733}x^{733} + \ln(x)$$