Solutions - Part I

1. For each function, find its derivative.

   (a) $k(x) = 3\log_4(x) + 2\sin(e^x) - \ln(x^2+1) + \tan^2(x)$

   $$k'(x) = 3\left(\frac{1}{\ln(4)}\right)\left(\frac{1}{x}\right) + 2\cos(e^x)(e^x) - \frac{1}{x^2+1}(2x) + 2\tan(x)(\sec^2(x)).$$

   (b) $l(x) = \log_2\left(e^x + 2\right)$

   $$l'(x) = \frac{1}{\ln(2)}\frac{1}{e^x+2}(e^x).$$

Solutions - Part I

1. For each function, find its derivative.

   (c) $m(x) = 4^x \tan(x)$

   $$m'(x) = 4^x\big(\sec^2(x)\big) + \big(\ln(4)4^x\big)\tan(x).$$

   (d) $n(x) = \dfrac{\sec(x) + x^2}{e^x - 4\cos(x)}$

   $$n'(x) = \frac{\big(e^x - 4\cos(x)\big)\cdot\big(\sec(x)\tan(x) + 2x\big) - \big(\sec(x) + x^2\big)\big(e^x + 4\sin(x)\big)}{\big(e^x - 4\cos(x)\big)^2}$$

   (e) $p(x) = (3x^2 - 7)\sin(5x + 3)$

   $$p'(x) = (3x^2 - 7)\big(\cos(5x+3)(5)\big) + (6x)\sin(5x+3).$$

Solutions - Part I

1. For each function, find its derivative.

(f) $r(x) = \dfrac{\sec(x^2)}{e^x}$

$$r'(x) = \frac{(e^x)\big(\sec(x^2)\tan(x^2)(2x)\big) - \sec(x^2)(e^x)}{(e^x)^2}$$

(g) $s(x) = x^5 e^x \sin(x^3)$

$$s'(x) = x^5 e^x \left(3x^2 \cos(x^3)\right) + x^5 \left(e^x\right) \sin(x^3) + \left(5x^4\right) e^x \sin(x^3)$$

(h) $t(x) = \sqrt{x^2 \cos(x)}$

$$t'(x) = \frac{1}{2}\left(x^2 \cos(x)\right)^{-1/2} \Big(x^2\left(-\sin(x)\right) + \left(2x\right)\cos(x)\Big)$$

Solutions - Part I

2. Find the *second* derivative of each of the following:

(a) $g(x) = \cos^2(x) + x^e - \ln(x) + 3^x$

$$\begin{aligned} f'(x) &= -2\cos(x)\sin(x) + ex^{e-1} - \frac{1}{x} + \ln(3)3^x \\ f''(x) &= \big(-2\cos(x)\cos(x) + 2\sin(x)\sin(x)\big) + e(e-1)x^{e-2} \\ &\quad + x^{-2} + \ln(3)\big(\ln(3)3^x\big) \\ &= 2\big(\sin^2(x) - \cos^2(x)\big) + e(e-1)x^{e-2} + \frac{1}{x^2} + \big(\ln(3)\big)^2 3^x \end{aligned}$$

(b) $j(x) = 4e^{2x} - 3\cos(x^2+1) - \frac{1}{x}$

$$\begin{aligned} j'(x) &= 4e^{2x}\cdot 2 + 3\sin(x^2+1)\cdot(2x) - (-1)x^{-2} \\ &= 8e^{2x} + 6x\sin(x^2+1) + x^{-2} \\ j''(x) &= 16e^{2x} + \big(6x\cos(x^2+1)(2x) + 6\sin(x^2+1)\big) - 2x^{-3} \end{aligned}$$

Solutions - Part I

3. Find an antiderivative of each of the following; check your answers by differentiating your result.

   (a) $f(x) = e^x - \frac{1}{x} + \sin(x)$

   $$F(x) = e^x - \ln(x) - \cos(x).$$

   Check: $\dfrac{d}{dx}\Big(e^x - \ln(x) - \cos(x)\Big) = e^x - \dfrac{1}{x} - \big(-\sin(x)\big)$

   (b) $g(x) = e^{x+2} - \frac{1}{x+3} + \sin(x+4)$

   $$G(x) = e^{x+2} - \ln(x+3) - \cos(x+4).$$

   Check:
   $\dfrac{d}{dx}\Big(e^{x+2} - \ln(x+3) - \cos(x+4)\Big) = e^{x+2}(1) - \dfrac{1}{x+3}(1) + \sin(x+4)(1)$

Solutions - Part I

3. Find an antiderivative of each of the following; check your answers by differentiating your result.

   (c) $h(x) = e^{2x} - \frac{1}{3x} + \sin(4x)$

$$H(x) = \frac{1}{2}e^{2x} - \frac{1}{3}\ln(3x) - \frac{1}{4}\cos(4x)$$

Check:

$$\frac{d}{dx}\left(\frac{1}{2}e^{2x} - \frac{1}{3}\ln(3x) - \frac{1}{4}\cos(4x)\right) = \frac{1}{2}(2)e^{2x} - \frac{1}{3}(3)\frac{1}{3x} + \frac{1}{4}(4)\sin(4x)$$

Part II

4. Suppose you know that $h(x) = f\big(g(x)\big)$, and you have also observed that $f(1) = 3$, $g(1) = 2$, $f'(1) = 4$, $f'(2) = 3$, $g'(1) = -2$, and $g'(3) = 5$. Find $h'(1)$.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\Big(f\big(g(x)\big)\Big) \\ &= f'\big(g(x)\big)g'(x) \\ \\ \Rightarrow h'(1) &= f'\big(g(1)\big)g'(1). \end{aligned}$$

Since $g(1) = 2$, $g'(1) = -2$,

$$\Rightarrow h'(1) = f'(2) \cdot (-2).$$

And since $f'(2) = 3$,

$$\Rightarrow h'(1) = (3)(-2) = -6$$

Part II

5. Find the derivative of $f(\sqrt{x})$, as far as you can for an unspecified function f.

$$\frac{d}{dx}\left(f\left(\sqrt{x}\right)\right) = f'\left(\sqrt{x}\right) \cdot \frac{1}{2}x^{-1/2}.$$

Part II

6. Find an antiderivative $F(x)$ of $f(x)$:

   (a) $f(x) = (x^2 + 3)^2(2x)$

   - What could I have differentiated to produce the product $(x^2 + 3)^2(2x)$?
   - Some sort of *product* or *composition*.
   - Differentiating a product usually produces a sum; I don't have a sum.
   - Differentiating a composition usually produces a product of 2 pieces, one of which is usually still a composition. Do I have a composition?
   - Yes - $(x^2 + 3)^2$
   - So I'm going to guess that $(x^2 + 3)^2(2x)$ comes from differentiating a composition using the chain rule.

Part II

6. Find an antiderivative $F(x)$ of $f(x)$:

   (a) $f(x) = (x^2+3)^2(2x)$ (continued)

   What $F(x) = h\big(g(x)\big)$ might I have started with to end up with
   $$h'\big(g(x)\big)g'(x) = (x^2+3)^2(2x)?$$

   - **Notice:** If $g(x) = x^2+3$, then $g'(x) = 2x$, so that seems reasonable.
   - Let $u = x^2+3$. Then what I'm left with is $h'(u) = u^2$. In that case, what is $h(u)$?
     $$h(u) = \frac{1}{3}u^3.$$
   - Thus I think $F(x) = h\big(g(x)\big) = h(x^2+3) = \frac{1}{3}(x^2+3)^3$.
   - **Check:** Is $F(x)$ an antiderivative of $f(x)$? That is, is $F'(x) = f(x)$?
     $$\frac{d}{dx}\left(\frac{1}{3}(x^2+3)^3\right) = \frac{1}{3}(3)(x^2+3)^2(2x) = (x^2+3)(2x) = f(x),$$ as desired

Part II

6. Find an antiderivative $F(x)$ of $f(x)$:

   (b) $f(x) = (x^2 + 3x)(2x) + (2x + 3)(x^2)$

   - What could I have differentiated to produce the product
     $$(x^2 + 3x)(2x) + (2x + 3)(x^2)?$$
   - Some sort of *product* or *composition*.
   - Differentiating a product usually produces a sum; I have a sum.
   - Differentiating a composition usually produces a product of 2 pieces, one of which is usually still a composition. I don't have a composition.
   - So I'm going to guess that $(x^2 + 3x)(2x) + (2x + 3)(x^2)$ comes from differentiating a product using the product rule.

Part II

6. Find an antiderivative $F(x)$ of $f(x)$:

   (b) $f(x) = (x^2 + 3x)(2x) + (2x + 3)(x^2)$

   What $F(x) = g(x)h(x)$ might I have started with to end up with
   $$g(x)h'(x) + g'(x)h(x) = (x^2 + 3x)(2x) + (2x + 3)(x^2)?$$

   - If I let $g(x) = x^2 + 3x$, then $g'(x) = 2x + 3$ which shows up in the other half of the expression.
   - If I let $h(x) = x^2$, then $h'(x) = 2x$, which shows up in the first half of the expression.
   - Thus I think $F(x) = g(x)h(x) = (x^2 + 3x)(x^2)$
   - **Check:** Is $F(x)$ an antiderivative of $f(x)$? That is, is $F'(x) = f(x)$?
     $$\frac{d}{dx}\Big((x^2 + 3x)(x^2)\Big) = (x^2 + 3x)(2x) + (2x + 3)(x^2) = f(x), \text{ as desired}$$

Part II

7. Over what intervals is $f(x) = (x^2-4)^2(x^2+10)^2$ increasing? Where is it decreasing? Use your answers to determine where f has local maxima and local minima.

- $f(x)$ is increasing whenever it's rate of change is positive, that is, wherever $f'(x) > 0$. It is decreasing wherever $f'(x) < 0$.
- f is a product of $(x^2-4)^2$ and $(x^2+10)^2$, so in order to find f', use the product rule:

$$f'(x) = (x^2-4)^2\Big((x^2+10)^2\Big)' + \Big((x^2-4)^2\Big)'(x^2+10).$$

- Each of the pieces that still need to be differentiated are compositions, and so next use the chain rule:

$$\begin{aligned} f'(x) &= (x^2-4)^2\Big[2(x^2+10)(2x)\Big] + \Big[2(x^2-4)(2x)\Big](x^2+10)^2 \\ &= 4x(x^2-4)^2(x^2+10) + 4x(x^2-4)(x^2+10)^2 \\ &= 4x(x^2-4)(x^2+10)\big[(x^2-4)+(x^2+10)\big] \\ &= 4x(x^2-4)(x^2+10)(2x^2+3) \end{aligned}$$

Part II

7. (continued) Over what intervals is $f(x) = (x^2 - 4)^2(x^2 + 10)^2$ increasing? Where is it decreasing? Use your answers to determine where f has local maxima and local minima.
   - To find where f' is positive and negative,
     - first find where it's 0.

$$\begin{aligned} f'(x) = 0 &\iff 4x(x^2 - 4)(x^2 + 10)(2x^2 + 3) = 0 \\ &\iff 4x = 0 \text{ or } x^2 - 4 = 0 \text{ or } x^2 + 10 = 0 \text{ or } 2x^2 + 3 = 0 \\ &\iff x = 0 \text{ or } x = \pm 2 \end{aligned}$$

     - These are the only places $f'(x) = 0$, and thus the only places $f'(x)$ could possibly go from being positive to negative or from negative to positive.
     - Thus if $f'(x) < 0$ (or > 0) at one point to the left of $x = -2$, it is negative (or positive) for *all* points to the left of $x = -2$.

       $x = -3$ is to the left of $x = -2$.

       $f'(\text{-}3) = (4 \cdot \text{-}3)\left((\text{-}3)^2 - 4\right)\left((\text{-}3)^2 + 10\right)\left(2 \cdot (\text{-}3)^2 + 3\right) = (\text{-})(+)(+)(+) < 0.$

       Since $f'(-3) < 0$, $f' < 0$ for all $x < -2$ and so f is decreasing on $(-\infty, -2)$.

Part II

7. (continued) Over what intervals is $f(x) = (x^2-4)^2(x^2+10)^2$ increasing? Where is it decreasing? Use your answers to determine where f has local maxima and local minima.

- To find where f' is positive and negative,
  - Just found f is decreasing on the interval $(-\infty, -2)$.
  - Since $f'(x) = 0$ at $x = -2$ and at $x = 0$ but at no points in between, $f'(x)$ must have the same sign for all x in the interval $(-2, 0)$.

    Pick one point in that interval: $x = -1$ is in the interval $(-2, 0)$.

    $f'(\text{-}1) = \left(4\cdot(\text{-}1)\right)\left((\text{-}1)^2 - 4\right)\left((\text{-}1)^2 + 10\right)\left(2(\text{-}1)^2 + 3\right) = (\text{-})(\text{-})(+)(+) > 0.$

    Since $f'(-1) > 0$, $f > 0$ for all x in $(-2, 0)$ and so f is increasing on $(-2, 0)$.

Part II

7. (continued) Over what intervals is $f(x) = (x^2 - 4)^2(x^2 + 10)^2$ increasing? Where is it decreasing? Use your answers to determine where f has local maxima and local minima.

- To find where f' is positive and negative,
  - f is decreasing on the interval $(-\infty, -2)$.
  - f is increasing on the interval $(-2, 0)$
  - To find out whether f' is positive or negative on $(0, 2)$, find the sign of f' at one point in that interval: try $x = 1$.
    $$f'(1) = \left(4 \cdot 1\right)\left(1^2 - 4\right)\left(1^2 + 10\right)\left(2(1)^2 + 3\right) = (+)(-)(+)(+) < 0.$$
    Thus $f' < 0$ on the interval $(0, 2)$ and so f is decreasing on $(0, 2)$.
  - And to find out whether f' is positive or negative for all $x > 2$, find the sign of f' at one point to the right of $x = 2$, such as $x = 3$.
    $$f'(3) = \left(4 \cdot 3\right)\left(3^2 - 4\right)\left(3^2 + 10\right)\left(2 \cdot 3^2 + 3\right) = (+)(+)(+)(+) > 0,$$
    so $f' > 0$ on $(2, \infty)$ and hence f is increasing on $(2, \infty)$.

Part II

7. (continued) Over what intervals is $f(x) = (x^2 - 4)^2(x^2 + 10)^2$ increasing? Where is it decreasing? Use your answers to determine where f has local maxima and local minima.

   - To find where f' is positive and negative,
     - f is decreasing on the interval $(-\infty, -2)$.
     - f is increasing on the interval $(-2, 0)$
     - f is decreasing on the interval $(0, 2)$
     - f is increasing on the interval $(2, \infty)$
     - Thus f has a local minimum at $x = -2$, and another one $x = 2$. It has a local maximum at $x = 0$.

Part III

8. Find a function which is flat at $x = -1$, $x = 0$, $x = 2$.

   Hint: Construct your function's derivative first.

   $f(x)$ will be flat at $x = -1$, $x = 0$, and $x = 2$ if:

   $$f'(-1) = 0 \qquad f'(0) = 0 \qquad f'(2) = 0.$$

   One example of such a derivative is

   $$f'(x) = (x+1)(x)(x-2).$$

   This multiplies out to

   $$f'(x) = x(x^2 - x - 2) = x^3 - x^2 - 2x.$$

   Antidifferentiating this will produce a function that is flat at $x = -1$, $x = 0$, and $x = 2$:

   $$f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \text{ is one such function}$$

Part III

9. Suppose that $F(x)$ is an antiderivative of $f(x) = x\sin(e^x)$. Find $F'(x)$.

   Saying that $F(x)$ is an antiderivative of $f(x)$ is equivalent to saying that $F(x)$ is a function we differentiate to get $f(x)$, or in other words, $F'(x) = f(x)$.

   Thus $F'(x) = x\sin(e^x)$.

10. Suppose that $F'(x) = G'(x)$ for all x. Is it possible that $F(x) = G(x) + x$?

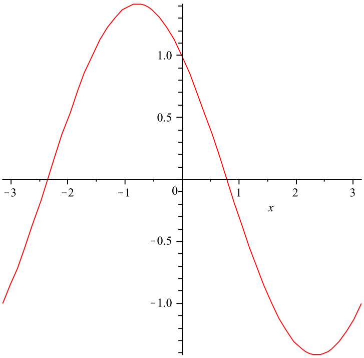
    Let's see if this is possible!

    Suppose $F(x)$ *is* $G(x) + x$. Then
    $F'(x) = \frac{d}{dx}(G(x) + x) = G'(x) + 1 \neq G'(x)$.

    So no, it is not possible that $F(x) = G(x) + x$.

Part III

11. Suppose that F is an antiderivative of the function f shown below. Suppose also that $F(0) = 10$.



(a) What is the slope of F at $x = 4$?

$$\text{Slope of } F \text{ at } x = 4 \;=\; F'(4)$$
$$= f(4) \approx 5.6.$$

(b) Where in the interval $[0, 10]$ is F increasing?

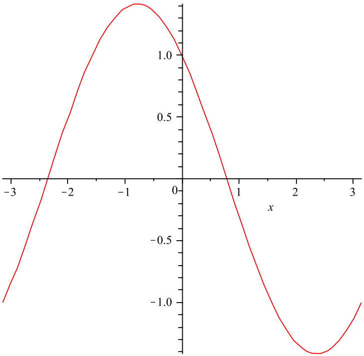
$F \uparrow$ where $F' > 0$, i.e. where $f > 0$, so on the entire interval $[0, 10]$

(c) Where in the interval $[0, 10]$ does F have inflection points?

F has inflection points where F' has local maxima or local minima, so at $x \approx 1.5$ and $x \approx 8.5$.

Part III

11. Suppose that F is an antiderivative of the function f shown below. Suppose also that $F(0) = 10$.



(d) Is it possible that $F(5) = 5$?

We know that $F(0) = 10$, and we know that F is increasing from $x = 0$ to $x = 10$, so $F(5)$ must be larger than $F(0)$.

Hence $F(5) > 10$, and so $F(5)$ can not possibly be 5.